Random variables were introduced as a function whose domain is the sample space of the random experiment and whose range is the real numbers. Random variables provide a method to unify the treatment of a wide variety of random experiments. Probability distribution and density functions were shown as fundamental methods of characterizing a random variable.

The study of functions of random variables led naturally to the concept of expectation and the statistical moments and covariance of random variables.

Gaussian random variables were introduced as a particular important type of random variable in the study of communication systems.

Considering time as parameter in random signals led to the study of random processes. A random process was defined as a family of random variables indexed by time as a parameter. Stationary, ergodic, and wide-sense stationary random processes were introduced as models of most physical processes exhibiting random behavior. It was shown that wide-sense stationary random processes have many of the properties of deterministic power signals, including the fact that the Weiner–Khintchine formulas relate the spectrum of the random process to its autocorrelation.

Gaussian processes and white noise were introduced as important random processes in the analysis of communication systems.

Finally, it was shown that, similar to deterministic signals, we may consider bandpass or narrowband versions of noise. This narrowband noise has in-phase and quadrature components, similar to deterministic signals.

This chapter has been a brief and certainly not complete introduction to the random signals and noise that are commonly found in communication systems, but the treatment presented herein is adequate for an introductory treatment of statistical communication theory. The two subsequent chapters will illustrate the importance of the material presented in this chapter in designing receivers and evaluating communication system performance.

Additional Problems

8.18 Consider a deck of 52 cards, divided into four different suits, with 13 cards in each suit ranging from the two up through the ace. Assume that all the cards are equally likely to be drawn.

(**a**) Suppose that a single card is drawn from a full deck. What is the probability that this card is the ace of diamonds? What is the probability that the single card drawn is an ace of any one of the four suits?

(**b**) Suppose that two cards are drawn from the full deck. What is the probability that the cards drawn are an ace and a king, not necessarily the same suit? What if they are of the same suit?

8.19 Suppose a player has one red die and one white die. How many outcomes are possible in the random experiment of tossing the two dice? Suppose the dice are indistinguishable, how many outcomes are possible?

8.20 Refer to Problem 8.19.

(**a**) What is the probability of throwing a red 5 and a white 2?

(**b**) If the dice are indistinguishable, what is the probability of throwing a sum of 7? If they are distinguishable, what is this probability?

8.21 Consider a random variable X that is uniformly distributed between the values of 0 and 1 with probability $\frac{1}{4}$, takes on the value 1 with probability $\frac{1}{4}$, and is uniformly distributed between values 1 and 2 with probability $\frac{1}{2}$. Determine the distribution function of the random variable X.

8.22 Consider a random variable X defined by the double-exponential density

$$f_X(x) = a \exp(-b|x|), \qquad -\infty < x < \infty$$

where a and b are constants.

(a) Determine the relationship between a and b so that $f_X(x)$ is a probability density function.

(b) Determine the corresponding distribution function $F_X(x)$.

(c) Find the probability that the random variable X lies between 1 and 2.

8.23 Show that the expression for the variance of a random variable can be expressed in terms of the first and second moments as

$$\mathrm{Var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2$$

8.24 A random variable R is Rayleigh distributed with its probability density function given by

$$f_R(r) = \begin{cases} \dfrac{r}{b}\exp\left(-\dfrac{r^2}{2b}\right), & 0 < r < \infty \\ 0, & \text{otherwise} \end{cases}$$

(a) Determine the corresponding distribution function $F_R(r)$.

(b) Show that the mean of R is equal to $\sqrt{b\pi/2}$.

(c) What is the mean-square value of R?

(d) What is the variance of R?

8.25 Consider a uniformly distributed random variable Z, defined by

$$f_Z(z) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le z \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

The two random variables X and Y are related to Z by $X = \sin(Z)$ and $Y = \cos(Z)$.

(a) Determine the probability density functions of X and Y.

(b) Show that X and Y are uncorrelated random variables.

(c) Are X and Y statistically independent? Why?

8.26 A Gaussian random variable has zero mean and a standard deviation of 10 V. A constant voltage of 5 V is added to this random variable.

(a) Determine the probability that a measurement of this composite signal yields a positive value.

(b) Determine the probability that the arithmetic mean of two independent measurements of this signal is positive.

8.27 Consider a random process $X(t)$ defined by

$$X(t) = \sin(2\pi Wt)$$

in which the frequency W is a random variable with the probability density function

$$f_W(w) = \begin{cases} \dfrac{1}{B}, & 0 \le w \le B \\ 0, & \text{otherwise} \end{cases}$$

Show that $X(t)$ is nonstationary.

8.28 Consider the sinusoidal process

$$X(t) = A\cos(2\pi f_c t)$$

where the frequency f_c is constant and the amplitude A is uniformly distributed:

$$f_A(a) = \begin{cases} 1, & 0 \le a \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Determine whether or not this process is stationary in the strict sense.

8.29 A random process $X(t)$ is defined by

$$X(t) = A\cos(2\pi f_c t)$$

where A is a Gaussian random variable of zero mean and variance σ_A^2. This random process is applied to an ideal integrator, producing an output $Y(t)$ defined by

$$Y(t) = \int_0^t X(\tau)\, d\tau$$

(**a**) Determine the probability density function of the output $Y(t)$ at a particular time t_k.

(**b**) Determine whether or not $Y(t)$ is stationary.

8.30 Prove the following two properties of the autocorrelation function $R_X(\tau)$ of a random process $X(t)$:

(**a**) If $X(t)$ contains a dc component equal to A, then $R_X(\tau)$ contains a constant component equal to A^2.

(**b**) If $X(t)$ contains a sinusoidal component, then $R_X(\tau)$ also contains a sinusoidal component of the same frequency.

8.31 A discrete-time random process $\{Y_n\}$ is defined by

$$Y_n = \alpha Y_n + W_n, \qquad n = \ldots, -1, 0, +1, \ldots$$

where the zero-mean random process $\{W_n\}$ is stationary with autocorrelation function $R_W(k) = \sigma^2\delta(k)$. What is the autocorrelation function $R_y(k)$ of Y_n? Is Y_n a wide-sense stationary process? Justify your answer.

8.32 Find the power spectral density of the process that has the autocorrelation function

$$R_X(\tau) = \begin{cases} \sigma^2(1 - |\tau|), & \text{for } |\tau| \le 1 \\ 0, & \text{for } |\tau| > 1 \end{cases}$$

8.33 A random pulse has amplitude A and duration T but starts at an arbitrary time t_0. That is, the random process is defined as

$$X(t) = A\,\mathrm{rect}(t + t_0)$$

where rect(t) is defined in Section 2.9. The random variable t_0 is assumed to be uniformly distributed over $[0, T]$ with density

$$f_{t_0}(s) = \begin{cases} \dfrac{1}{T}, & 0 \le s \le T \\ 0, & \text{otherwise} \end{cases}$$

(**a**) What is the autocorrelation function of the random process $X(t)$?

(**b**) What is the spectrum of the random process $X(t)$?

8.34 Given that a stationary random process $X(t)$ has an autocorrelation function $R_X(\tau)$ and a power spectral density $S_X(f)$, show that:

(**a**) The autocorrelation function of $dx(t)/dt$, the first derivative of $X(t)$, is equal to the negative of the second derivative of $R_X(\tau)$.

(**b**) The power spectral density of $dX(t)/dt$, is equal to $4\pi^2 f^2 S_X(f)$.
Hint: See the solution to problem 2.24