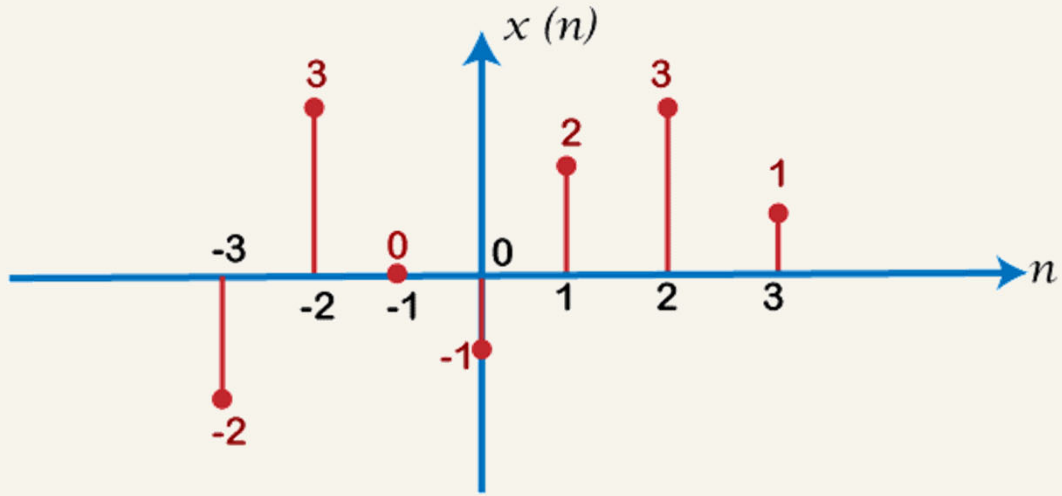
離散時間訊號與系統

Chapter 9
Discrete-Time Signals and Systems



- *A* discrete-time signal
  - is defined only at discrete instants of time.
- We denote a discrete-time signal as x[n],
  - where the independent variable *n* may assume only integer values.
- A discrete-time system
  - is defined as one in which all signals are discrete-time.
- Discrete-time Signal Processing (DSP)
- Sampling
  - Continuous-time functions → Discrete-times samples

- If the signal is sampled at regular increments of time T, the number sequence $f\ (nT), n\ = \cdots, -2, -1, 0, 1, 2, \ldots$, results.
T : **the *sampling period*.**

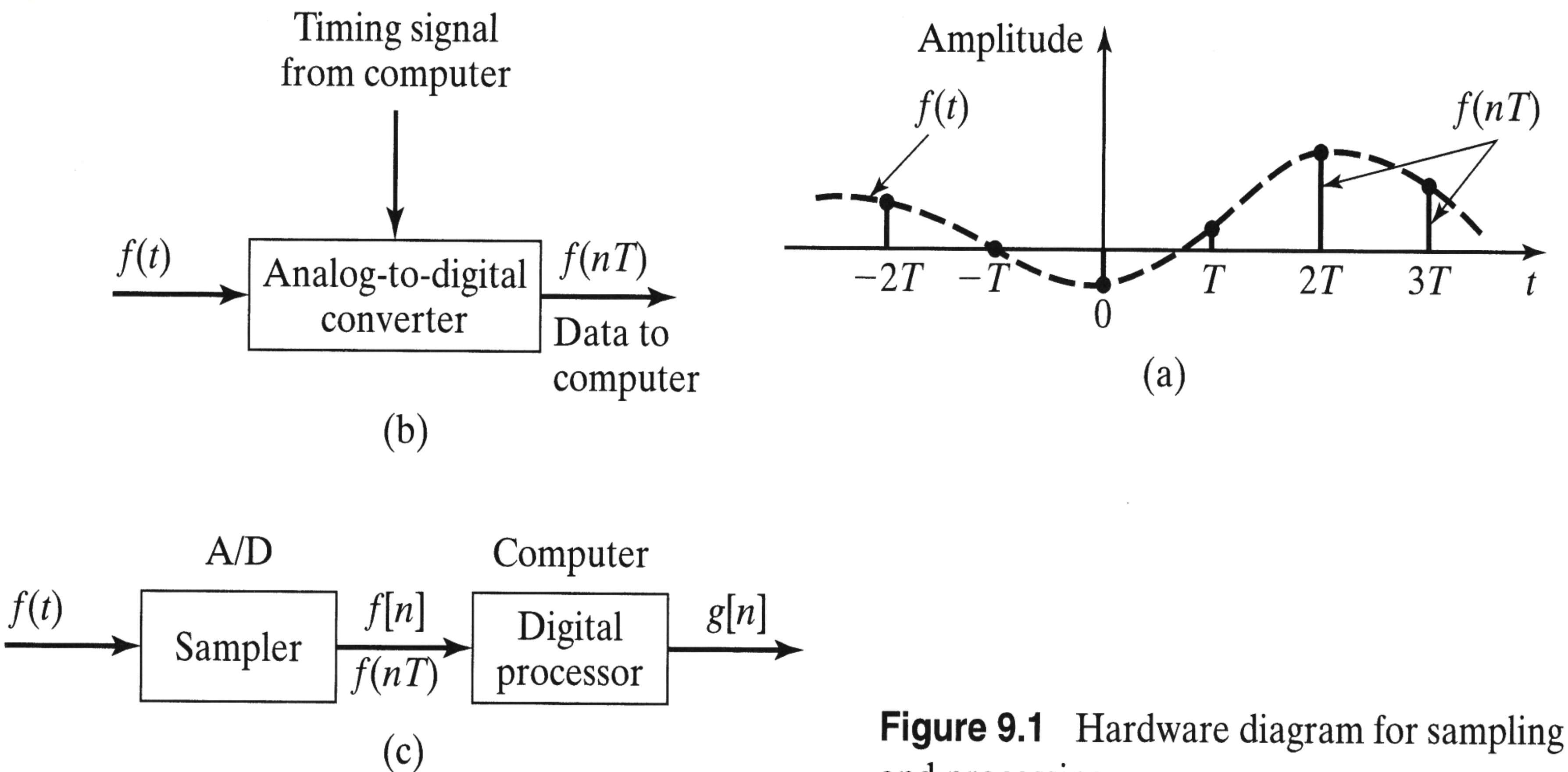


Figure 9.1 Hardware diagram for sampling and processing.

Notations

- $f(t)$: a continuous-time signal
- $f(nT)$: the value of $f(t)$ at $t = nT$
- $f[n]$: a discrete-time signal that is defined only for n an integer
- Parentheses ( ) : continuous time
- Brackets [ ] : discrete time.

Continuous-amplitude signal & Discrete-amplitude signal

$$f(nT) = f(t)\Big|_{t=nT}$$

$$f[n] = f(t)\Big|_{t=nT} \neq f(t)\Big|_{t=n}$$

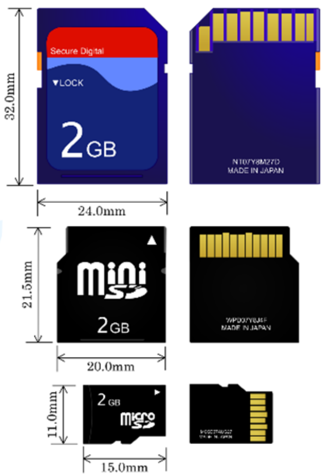
- A discrete-time signal can be a amplitude-continuous signal
- Discrete-amplitude signal: $x[n]$ can be defined only certain defined amplitude

→ Digital signal

Reasons that engineers are interested in discrete-time signals (Why Digital format?)

- Sampling is required if we are to use digital signal processing (DSP)
- Many communication systems are based on the transmission of discrete-time signals
- Sampling a signal allows us to store the signal in discrete memory.
- Automatically controlling physical systems require digital-computer implementation.
- Consumer products such as CDs, DVDs, digital cameras, and MP3 players use digital signals.

Blu-ray Disc™



Sec. 9.1 Discrete-Time Signals and Systems

- numerical integration as an example

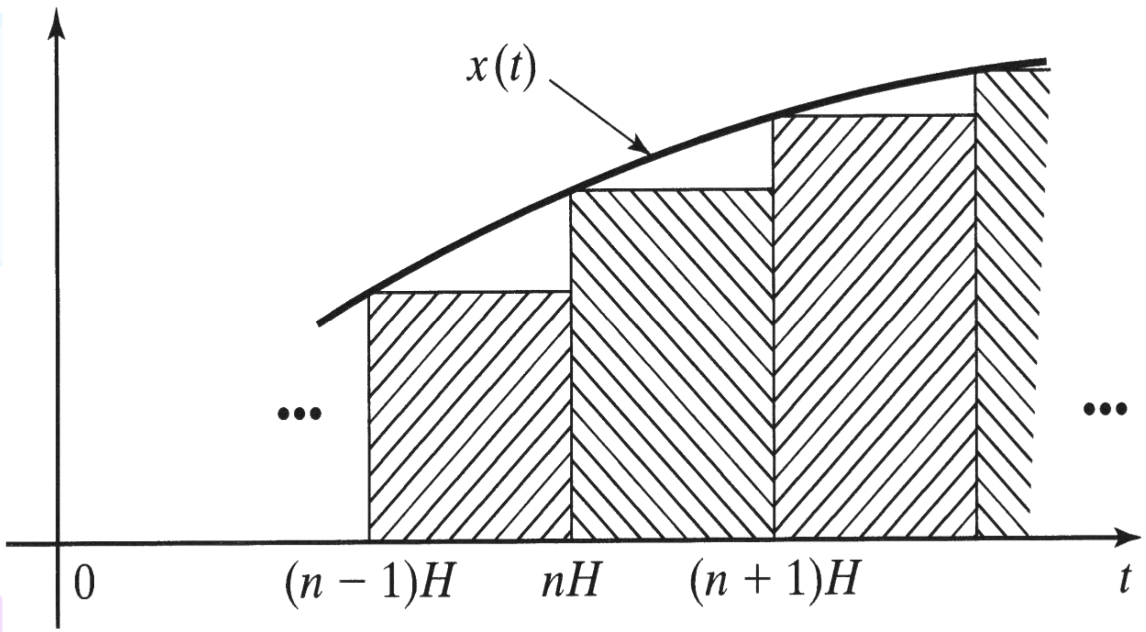


Figure 9.2 Euler integration.

A Difference Equation

$$y(t) = \int_0^t x(\tau) d\tau$$

$$y(t)\Big|_{t=nH} = y(nH) = \int_0^{nH} x(\tau)\, d\tau$$

$$= \int_0^{(n-1)H} x(\tau)\, d\tau$$

$$\approx y\big((n-1)H\big) + Hx\big((n-1)H\big)$$

Ignore $\approx$

$$\rightarrow \quad y(nH) = y\big((n-1)H\big) + Hx\big((n-1)H\big)$$

$$y[n] = y[n-1] + Hx[n-1]$$

Example 9.1
Difference-Equation solution

- Consider the numerical integration of a ***unit step function*** $u(t)$ using Eluer's rule.
- Assume initial condition $y(0) = 0$.

$x(nH) = 1$ for $n \geq 0$
$\rightarrow x[n] = 1$ for $n \geq 0$

$$y[n] = y[n-1] + Hx[n-1]$$

$$y[1] = y[0] + Hx[0] = 0 + H$$
$$y[2] = y[1] + Hx[1] = H + H$$
$$y[3] = y[2] + Hx[2] = 2H + H$$
$$\vdots$$
$$y[n] = y[n-1] + Hx[n-1]$$
$$= (n-1)H + H = nH$$

The exact integral of the unit step function

$$y(t) = \int_0^t u(\tau)d\tau = t\,, t > 0$$
$$y[n] = nH = y(t)\Big|_{t=nH}$$

Euler's rule gives the exact value for the integral of the unit step function

Unit Step Functions

- Discrete-time unit step function

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

- Time-shifted version

$$u[n - n_0] = \begin{cases} 1, & n \geq n_0 \\ 0, & n < n_0 \end{cases}$$

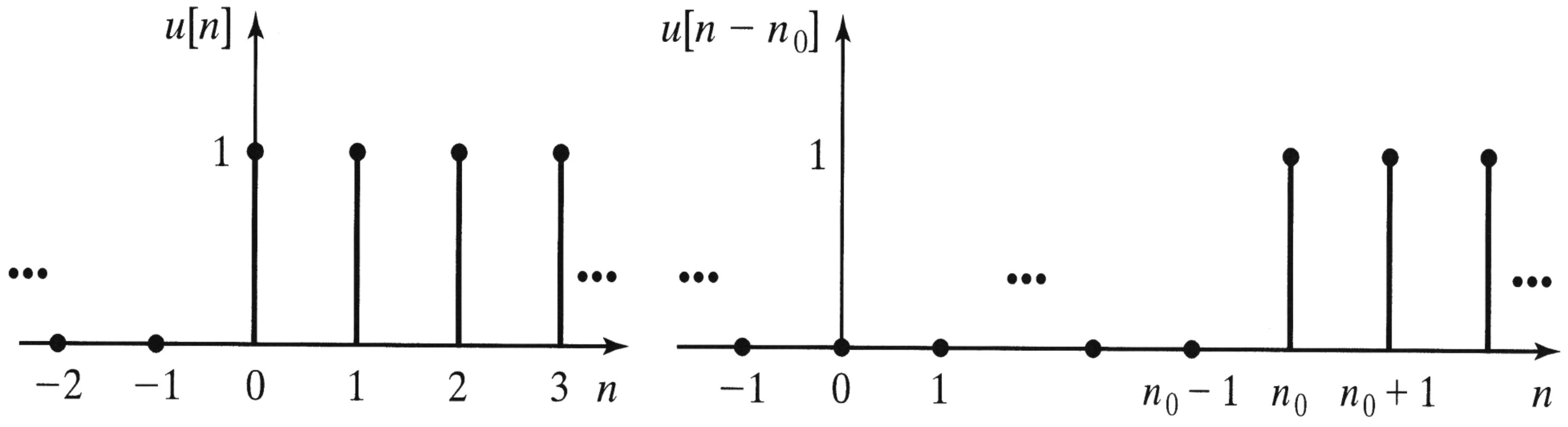


Figure 9.3 Discrete-time unit step functions.

Unit Impulse Functions

- Discrete-time unit Impulse function

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$\delta[n] = u[n] - u[n-1]$$

- Time-shifted version

$$\delta[n - n_0] = \begin{cases} 1, & n = n_0 \\ 0, & n \neq n_0 \end{cases}$$

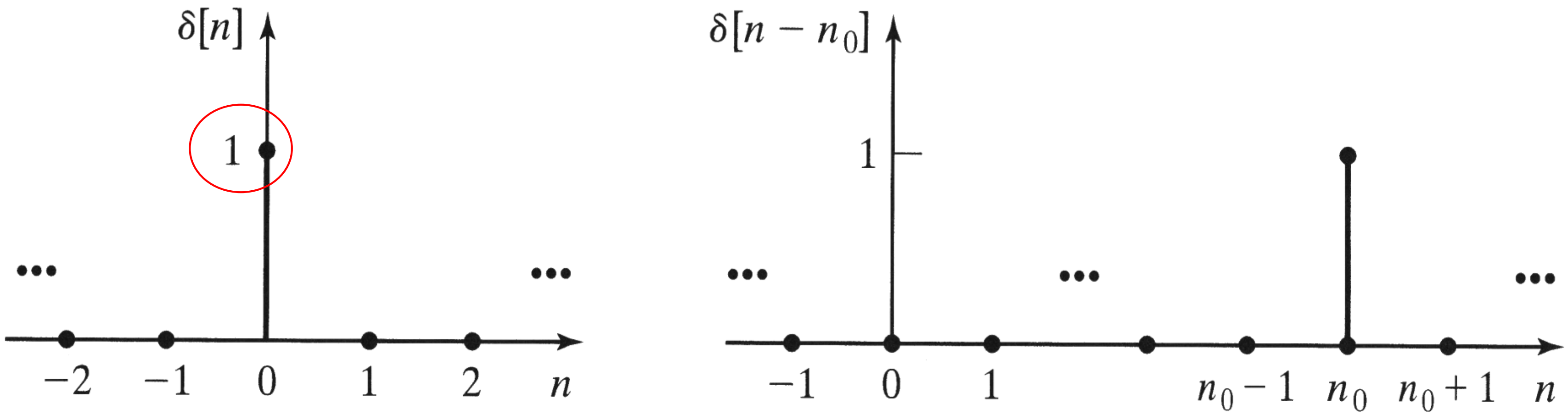


Figure 9.4 Discrete-time unit impulse functions.

Equivalent Operation

- integration in continuous time ⟺ summation in discrete time

By Euler's rule

$$\left.\int_{-\infty}^{t} x(\tau)\, d\tau\right|_{t=nH} \approx H \sum_{k=-\infty}^{n} x[k]$$

$$\int_{-\infty}^{t} x(\tau) d\tau \Leftrightarrow \sum_{k=-\infty}^{n} x[k]$$

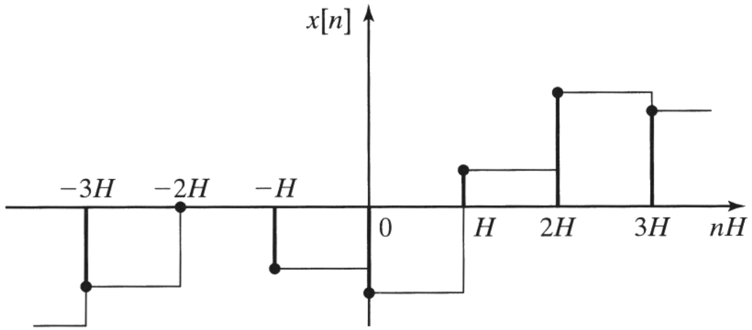


Figure 9.5 Summation yielding approximate integration.

- the first difference

$$\left.\frac{dx(t)}{dt}\right|_{t=kH} \approx \frac{x[k]-x[k-1]}{H}$$

$$\frac{dx(t)}{dt} \Leftrightarrow x[n]-x[n-1]$$

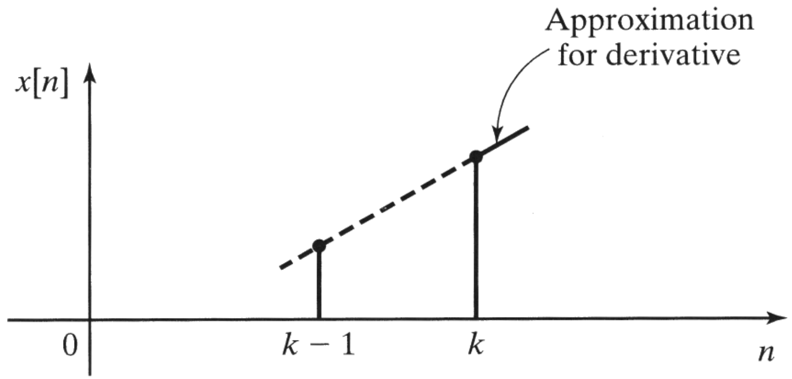


Figure 9.6 Approximate differentiation.

Equivalent Operation

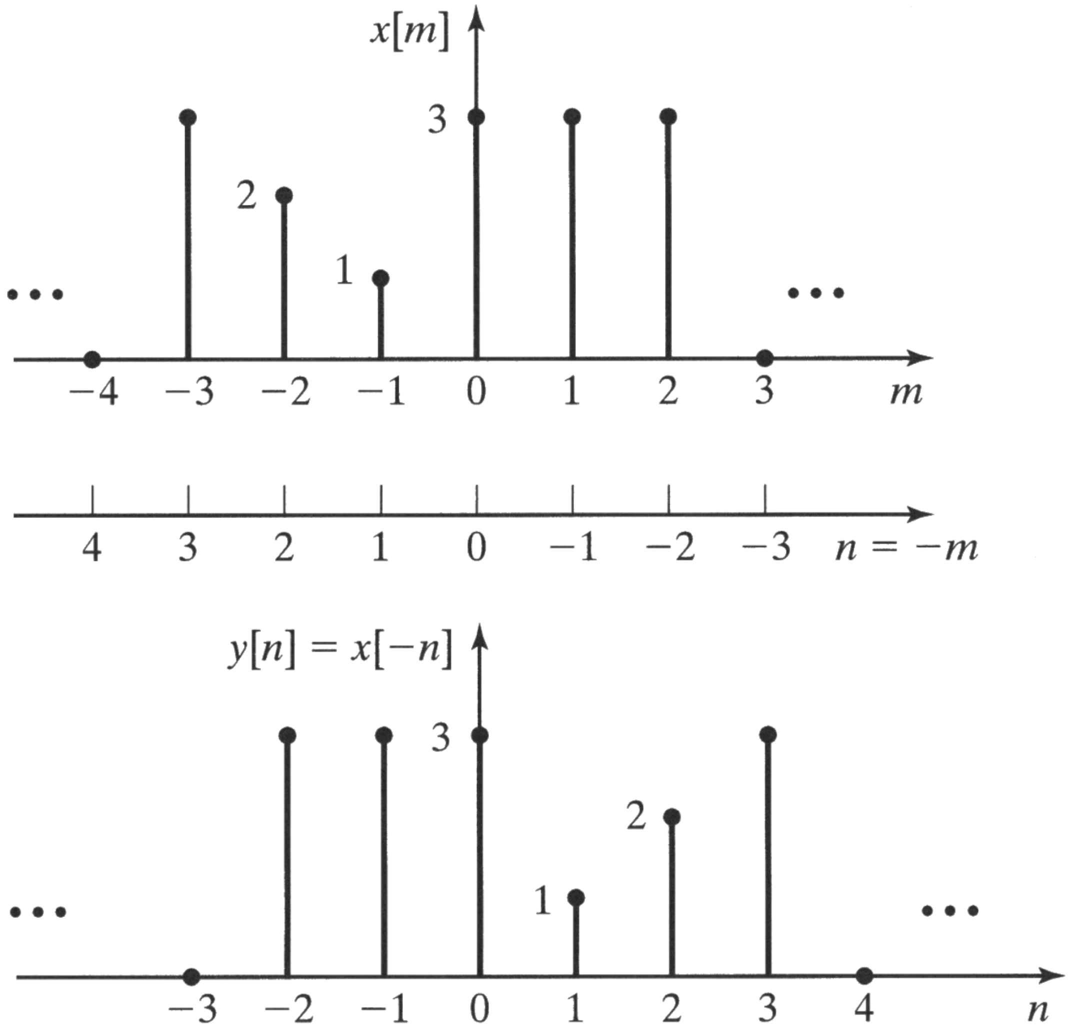
TABLE 9.1 Equivalent Operations

	Continuous Time	Discrete Time
1.	$\int_{-\infty}^{t} x(\tau)d\tau$	$\sum_{k=-\infty}^{n} x[k]$
2.	$\frac{dx(t)}{dt}$	$x[n] - x[n-1]$
3.	$x(t)\delta(t) = x(0)\delta(t)$	$x[n]\delta[n] = x[0]\delta[n]$
4.	$\delta(t) = \frac{du(t)}{dt}$	$\delta[n] = u[n] - u[n-1]$
5.	$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$	$u[n] = \sum_{k=-\infty}^{n} \delta[k]$

Time transformations

Time Reversal

$$y[n] = x[m]\Big|_{m=-n} = x[-n]$$



Time scaling

$$y[n] = x[m]\big|_{m=an} = x[an]$$

Down sampling

捨棄非整數 n 部分

$y_1[n] = x[2n]$

$y_1[n] = x[n/3]$

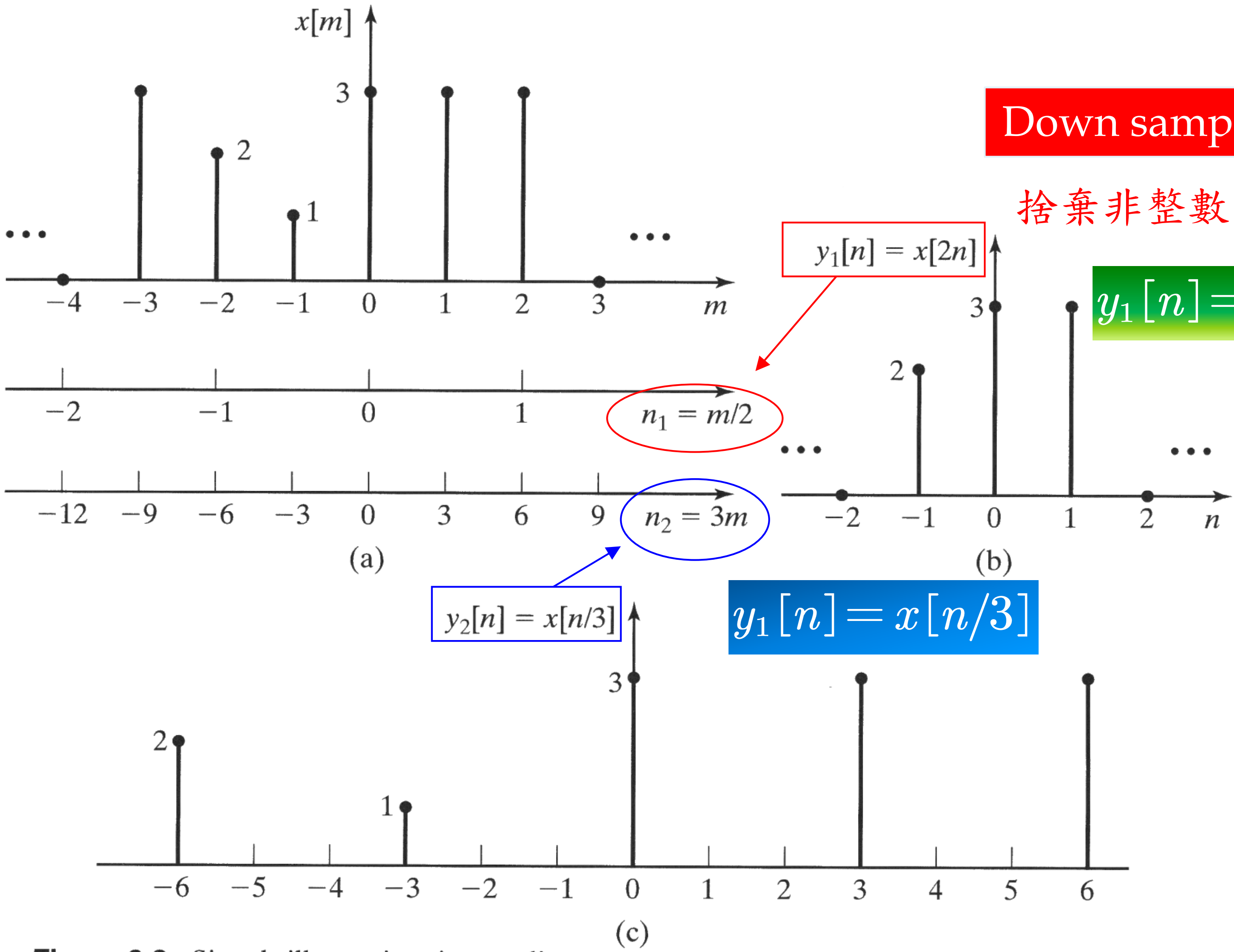


Figure 9.8 Signals illustrating time scaling.

Time scaling

Down sampling

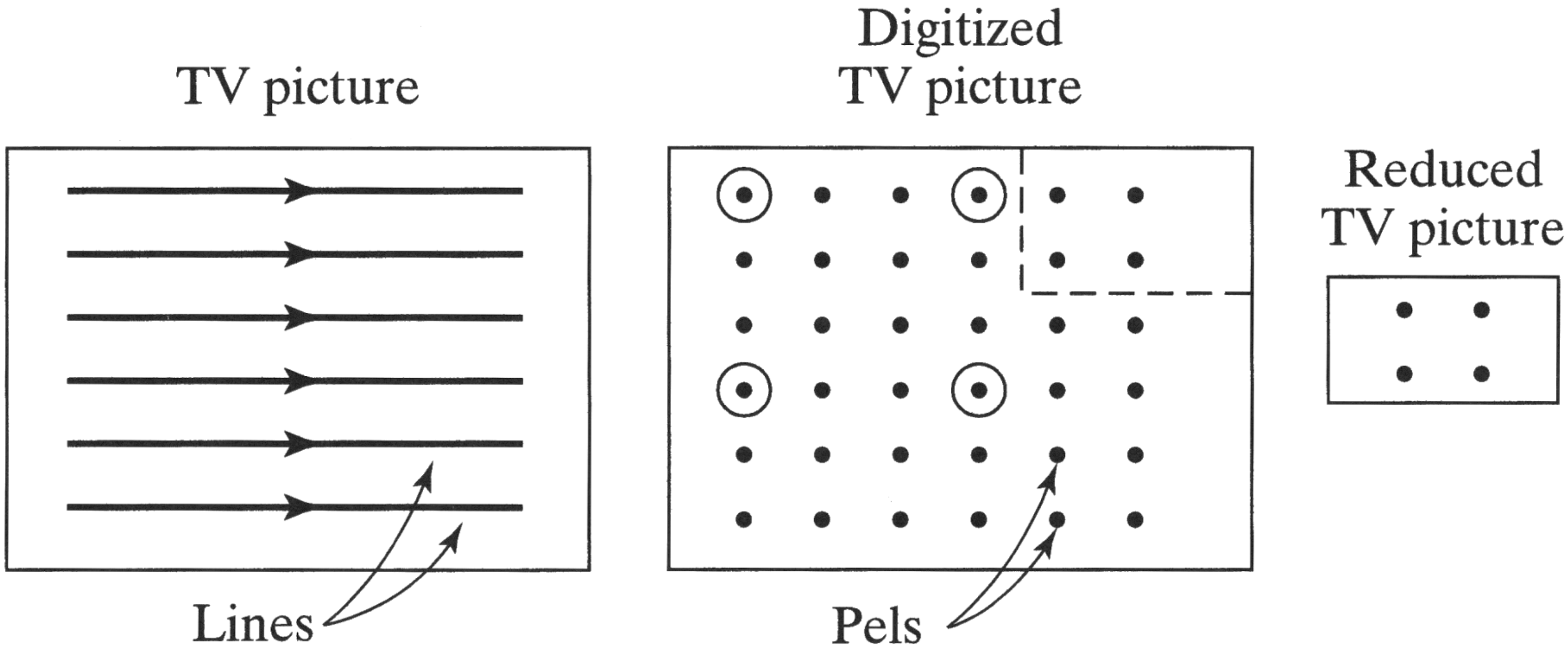


Figure 9.9 Television picture within a picture.

Time Shifting

$$y[n] = x[n - n_0]$$

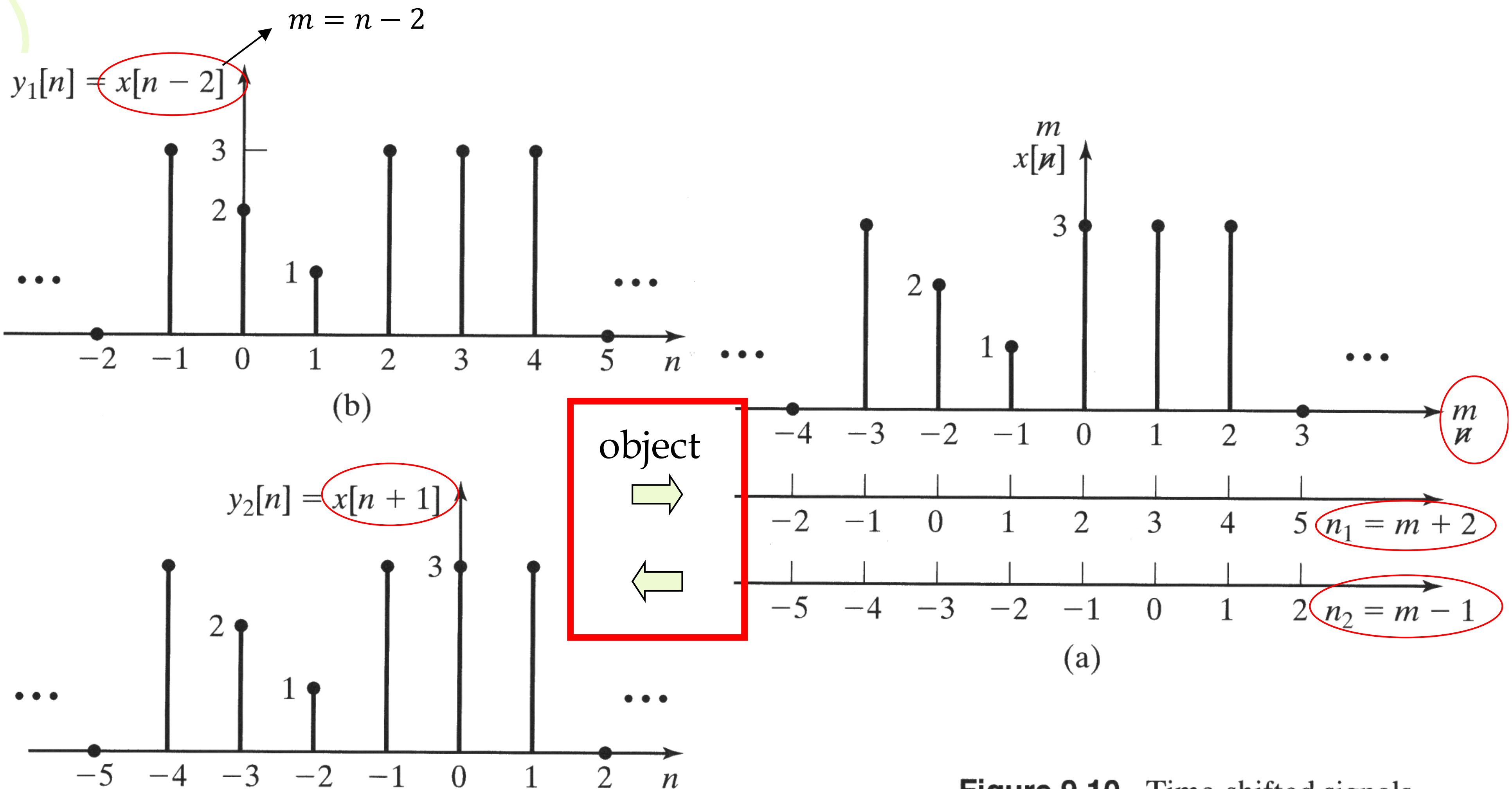


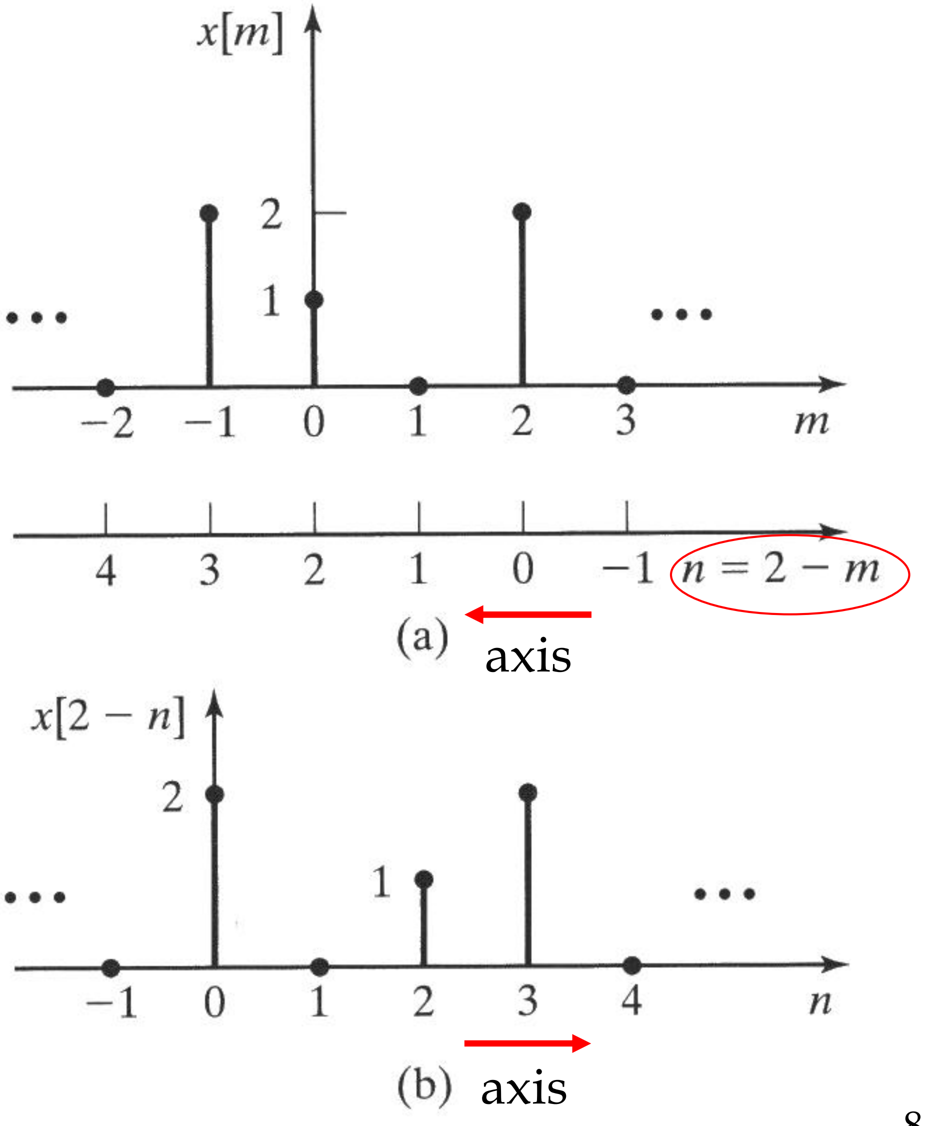
Figure 9.10 Time-shifted signals.

General Form $y[n] = x[an + b]$

Ex. 9-2

$y[n] = x[2 - n]$

$m = 2 - n \Rightarrow n = 2 - m$



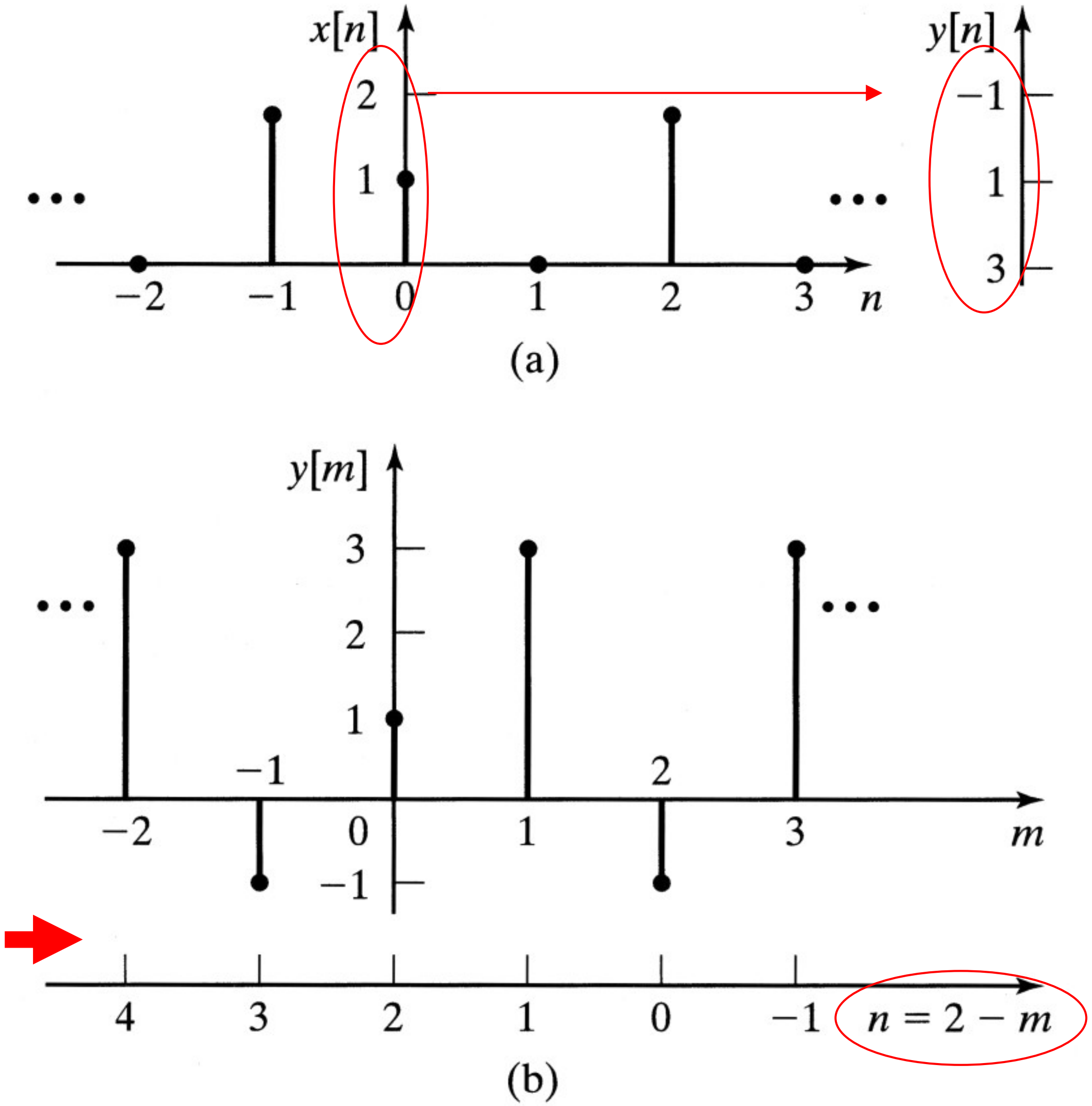
Amplitude transformation

$y[n] = Ax[n] + B$

Ex. 9.3

$y[n] = 3 - 2x[n]$

$x[n]$	$y[n]$
2 →	-1
1 →	1
0 →	3



Ex. 9.4

$y[n] = 3 - 2x[2 - n]$

Ex. 9.4 $y[n] = 3 - 2x[2 - n]$

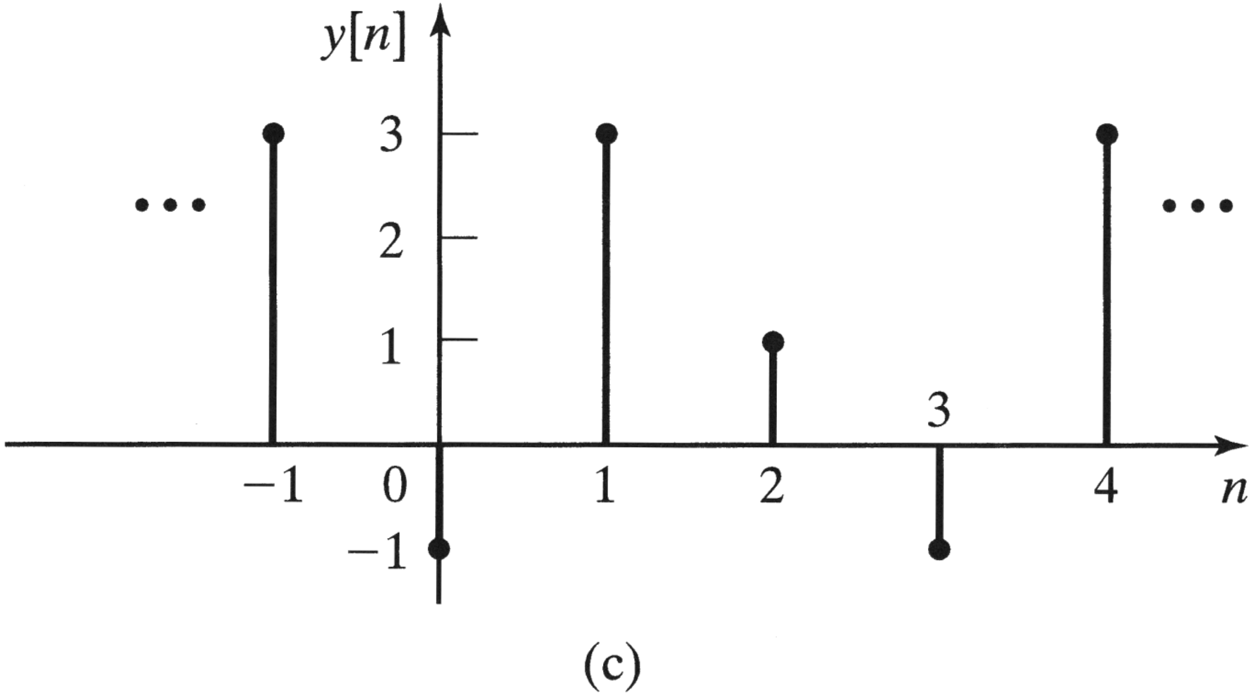


Figure 9.12 Signals for Examples 9.3 and 9.4.

Transformation of signals

TABLE 9.4 Transformations of Signals

Name	$y[n]$
Time reversal	$x[-n]$
Time scaling	$x[an]$
Time shifting	$x[n - n_0]$
Amplitude reversal	$-x[n]$
Amplitude scaling	$\|A\|x[n]$
Amplitude shifting	$x[n] + B$

9.3 Characteristics of discrete-time signals

Even and Odd Signals

even $x_e[n] = x_e[-n]$
odd $x_o[n] = -x_o[-n]$
$x[n] = x_e[n] + x_o[n]$

$$x_e[n] = \frac{1}{2}[x[n] + x[-n]]$$
$$x_o[n] = \frac{1}{2}[x[n] - x[-n]]$$

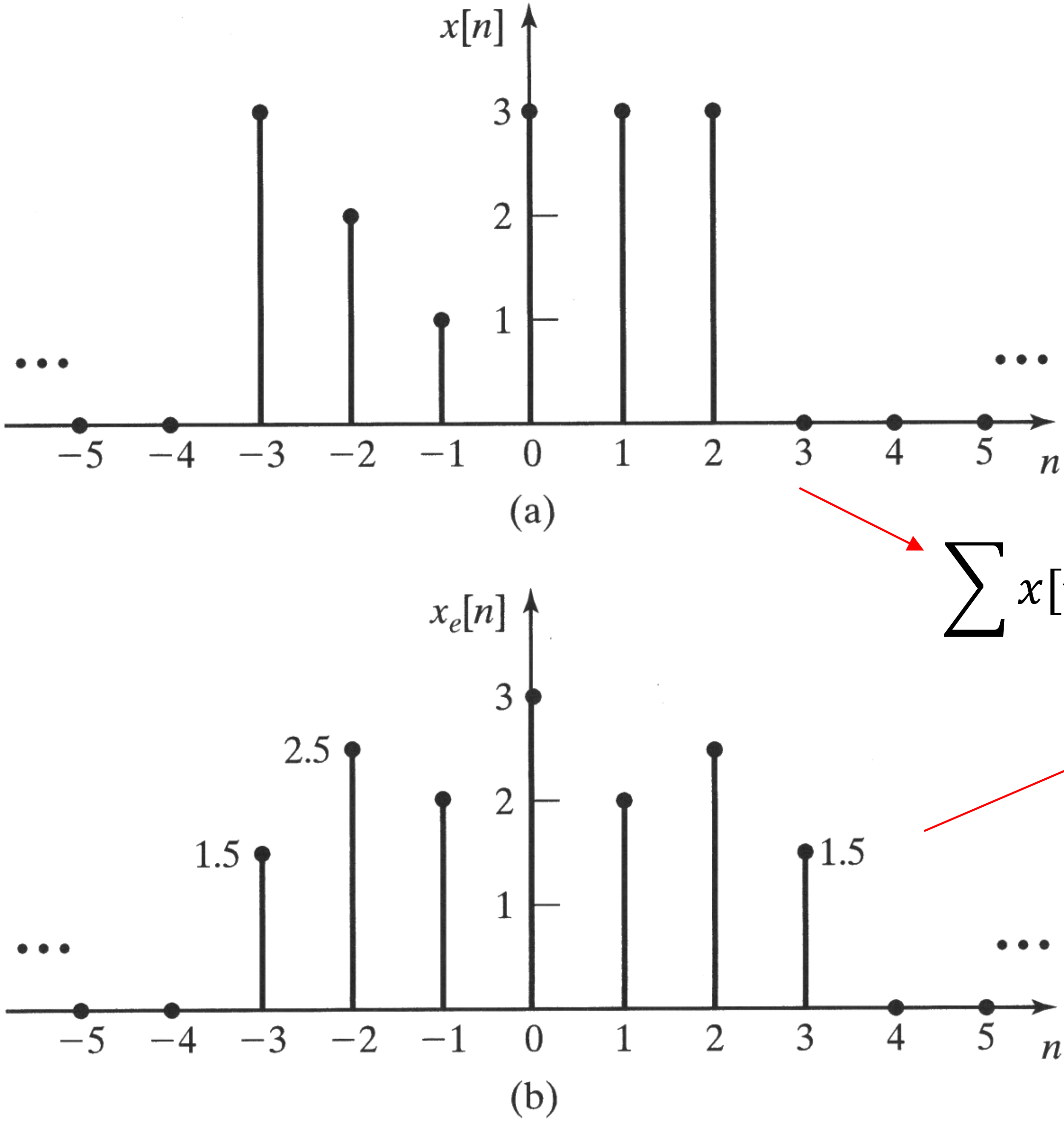
The average value, or mean value, of a discrete-time signal

$$A_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} x[k]$$

Properties of Even and odd signals

- The sum of two even signals is even.
- The sum of two odd signals is odd.
- The sum of an even signal and an odd signal is neither even nor odd.
- The product of two even signals is even.
- The product of two odd signals is even.
- The product of an even signal and an odd signal is odd.

Ex. 9.5 Even and Odd Functions



$$\sum x[n] = \sum x_e[n] = 15$$

Ex. 9.5 Even and Odd Functions

$$x_o[n] + x_o[-n] = x_o[n] - x_o[n] = 0$$

總和為 0

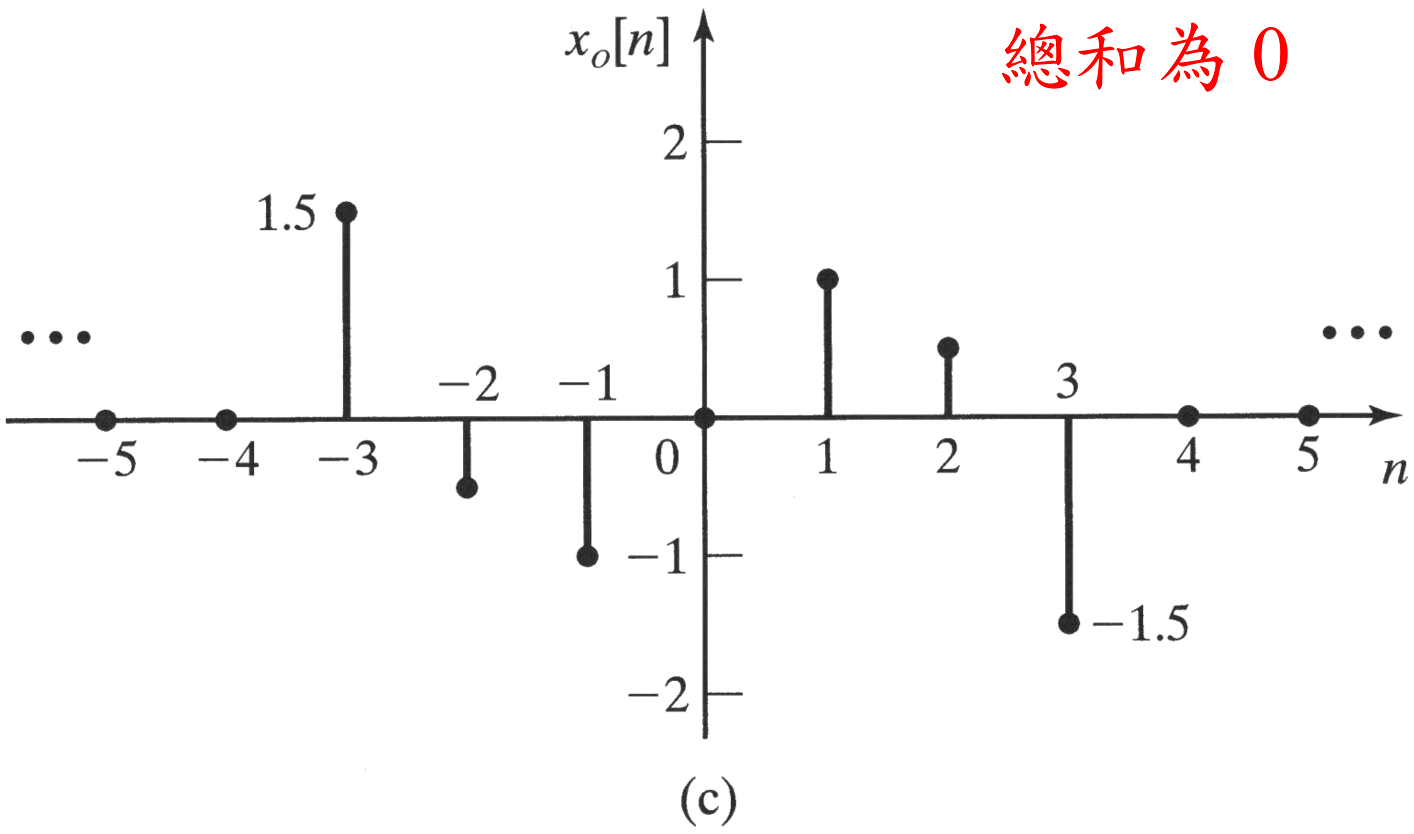


Figure 9.13 Signals for Example 9.5.

Signal Periodic in n

- A discrete-time signal is periodic with **period = N**

$$x[n+N] = x[n]$$

- Consider $x[n]$ is obtained by sampling a sinusoidal signal $x(t) = \cos(n\omega_0 t)$ every T sec.

$$x[n] = \cos(n\omega_0 T)$$
$$= x[n+N] = \cos[(n+N)\omega_0 T]$$
$$= \cos(n\omega_0 T + N\omega_0 T)$$

Hence, $N\omega_o T = N\frac{2\pi}{T_0}T = 2\pi k$, where k is any integer

$$\Rightarrow \frac{T}{T_0} = \frac{k}{N} \Rightarrow NT = kT_0$$, N samples in k periods of the signal

有理數

Ex. 9.6 Sampling of a sinusoid

$x(t) = \sin(\pi t) \Rightarrow T_0 = 2s$

$T = 0.5s \Rightarrow 4T = T_0$

$\frac{T}{T_0} = \frac{1}{4}$

1週期取4點

$T = 0.75s \Rightarrow 8T = 3T_0$

3週期取8點

5週期取4點

$T = 2.5s, T_0 = 2s \Rightarrow 4T = 5T_0$

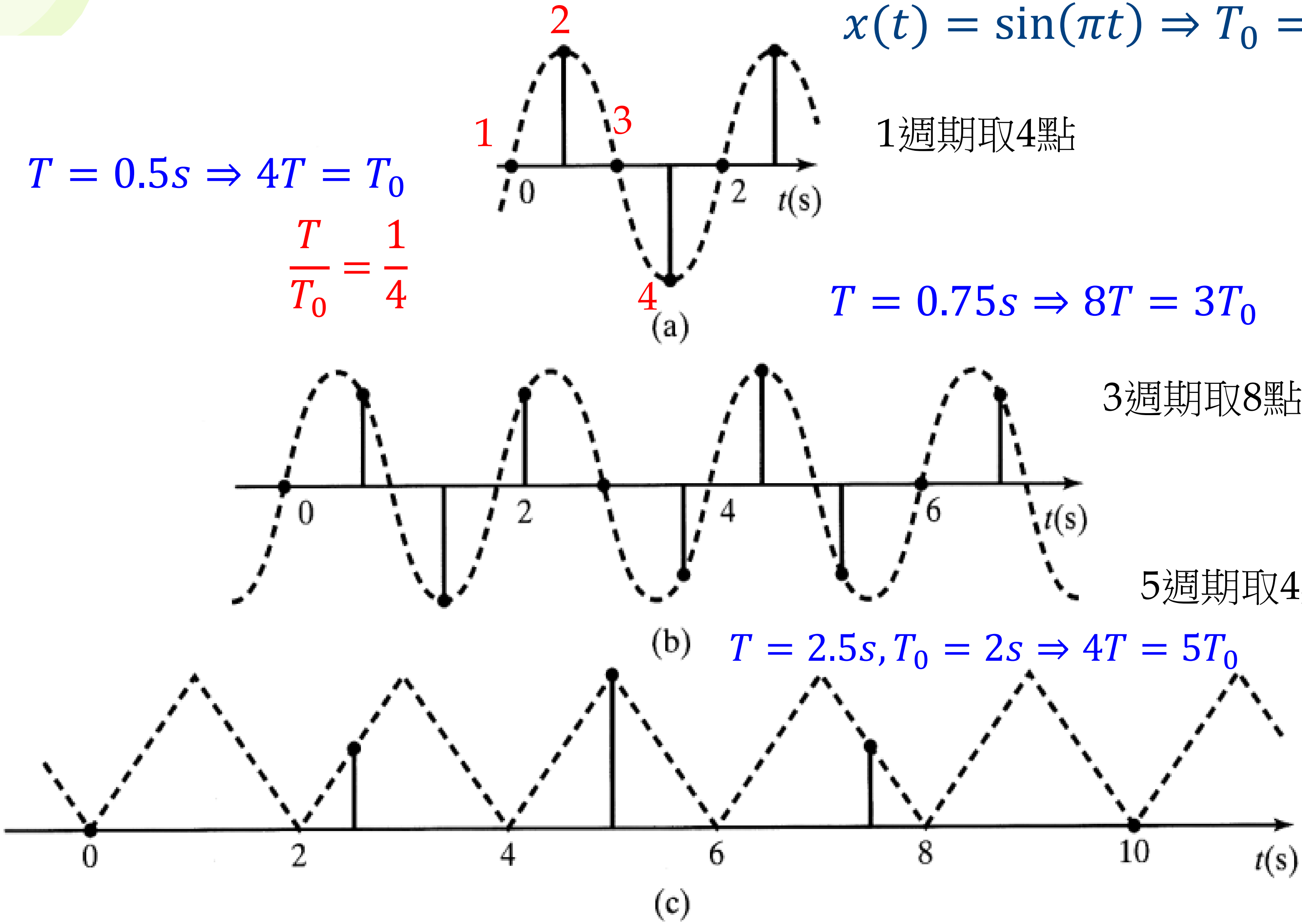


Figure 9.14 Periodic signals for Example 9.6.

Sampling of complex exponential signal

$$e^{j\omega_0 t}\Big|_{t=nT} = e^{j\omega_0 nT} = x[n]$$

$$\begin{aligned} e^{jn\omega_0 T} &= x[n] = x[n+N] \\ &= e^{j(n+N)\omega_0 T} \\ &= e^{jn\omega_0 T} e^{jN\omega_0 T} \end{aligned}$$

$$e^{jN\omega_0 T} = 1 = e^{j2\pi k}$$

$$N\omega_0 T = 2\pi k \Rightarrow NT = kT_0$$

$$\Rightarrow \frac{k}{N} = \frac{T_0}{T}$$

■ The discrete-time exponential signal that is not necessarily obtained by sampling a continuous-time signal.

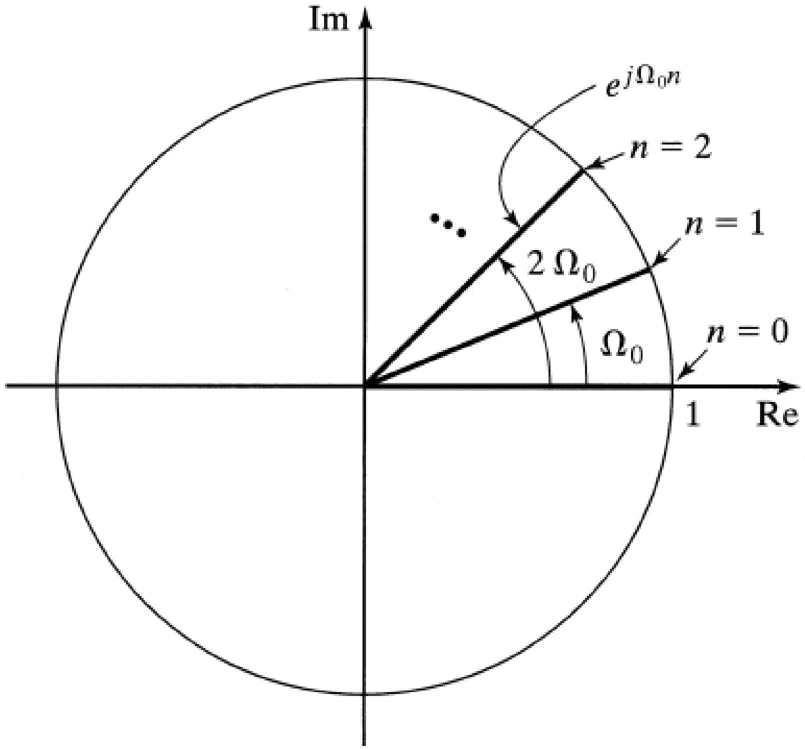
$$x[n] = e^{jn\omega_0 T} = e^{j\Omega_0 n} = 1\angle(\Omega_0 n)$$

$$\Omega_0 = \omega_0 T$$

正規化角頻率

$$x[n] = e^{j\Omega_0 n} = x[n+N]$$
$$= e^{j(\Omega_0 n + \Omega_0 N)}$$
$$= e^{j\Omega_0 n} e^{j2\pi k}$$

$$\Omega_0 N = 2\pi k \Rightarrow \Omega_0 = \frac{k}{N} 2\pi$$



Real frequency ω_0

Normalized discrete frequency $\Omega_0 = \omega_0 T$

取樣週期

Examples:

$$\Omega_0 = \frac{k}{N} 2\pi$$

$x[n] = \cos(2n)$

$$\Omega_0 = 2, \frac{k}{N} = \frac{1}{\pi} \notin \mathbf{Q}$$ ⟶ not periodic

$x[n] = \cos(0.1\pi n)$

$$\Omega_0 = 0.1\pi, \frac{k}{N} = \frac{1}{20} \in \mathbf{Q}$$ ⟶ periodic

(每個連續時間訊號週期，取幾個樣本)

N samples/periodic

$$N = \frac{2\pi k}{\Omega_0}\ ,\ k \text{ is the smallest integer}$$

Ex: $x[n] = \cos(0.1\pi n)$

$$N = \frac{2\pi k}{0.1\pi} = 20k, k = 1$$

$\Rightarrow N = 20$ samples/period

N samples/periodic

$N = \frac{2\pi k}{\Omega_0}$, k is the smallest integer

Ex: $x[n] = \cos(5\pi n)$

$$N = \frac{2\pi k}{5\pi} = 0.4k, k = 5$$

$$\Rightarrow N = 2$$

The discrete-time sinusoid $\cos(\Omega_0 n)$

1. $\cos(\Omega_0 n)$ is periodic in n only if,

$$\frac{\Omega_0}{2\pi} = \frac{k}{N}$$

2. $\cos(\Omega_0 n)$ is periodic in $\boldsymbol{\Omega}$ with periodic 2π

$$\cos(\Omega_0 n) = \cos(\Omega_0 + 2\pi k)\, n$$

9.4 COMMON DISCRETE-TIME SIGNALS

- Some equivalent discrete-time signals are introduced
- These signals can appear in the transient response of certain discrete-time systems.

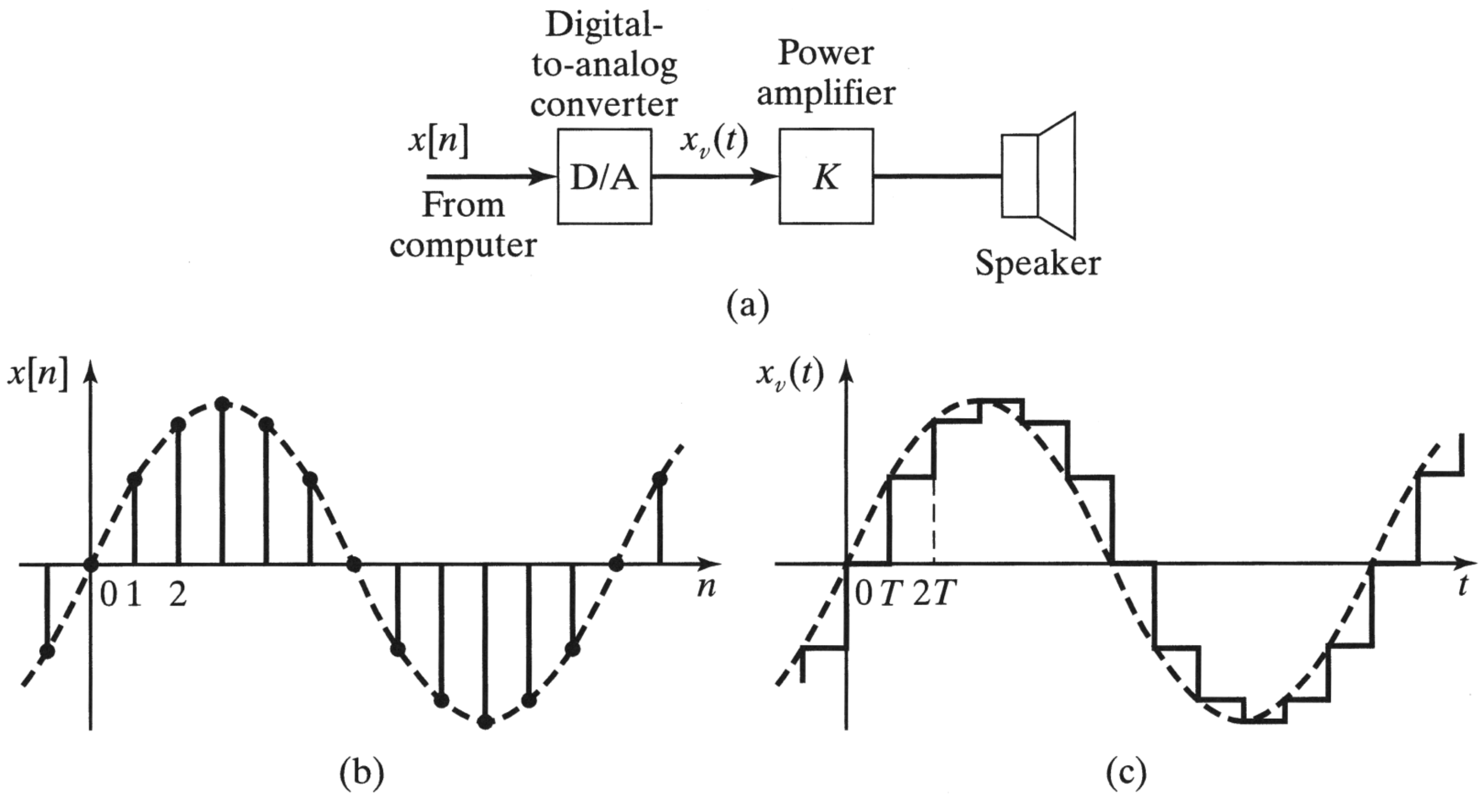


Figure 9.16 Computer generation of a tone.

COMMON DISCRETE-TIME SIGNALS

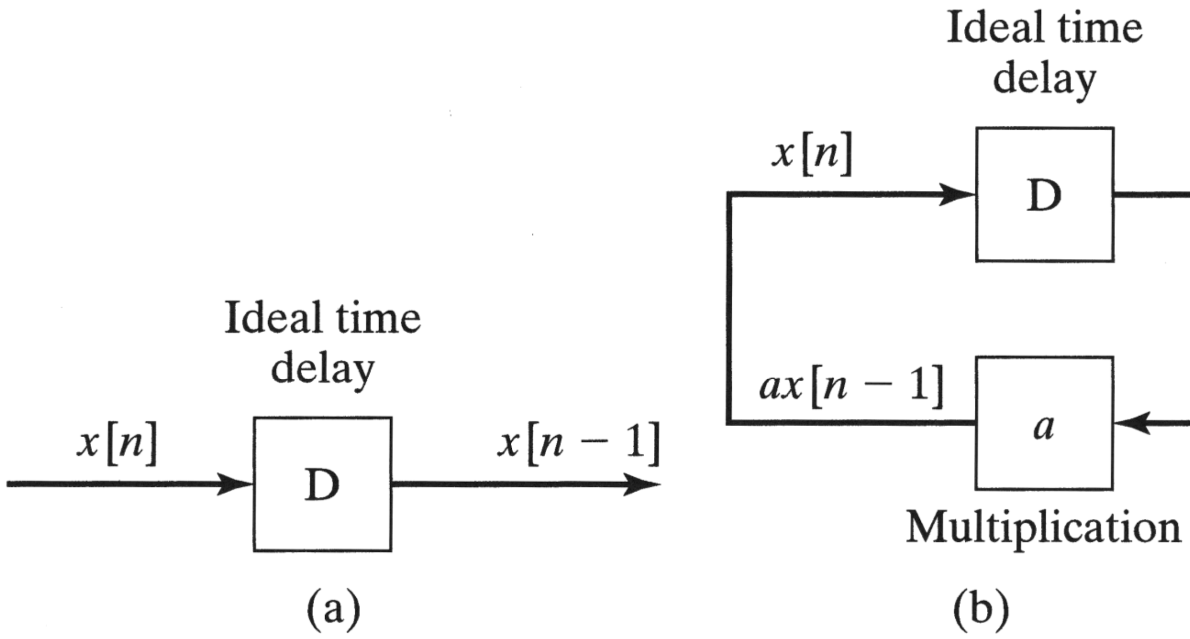


Figure 9.17 Discrete-time system.

$$x[n] = ax[n-1]$$

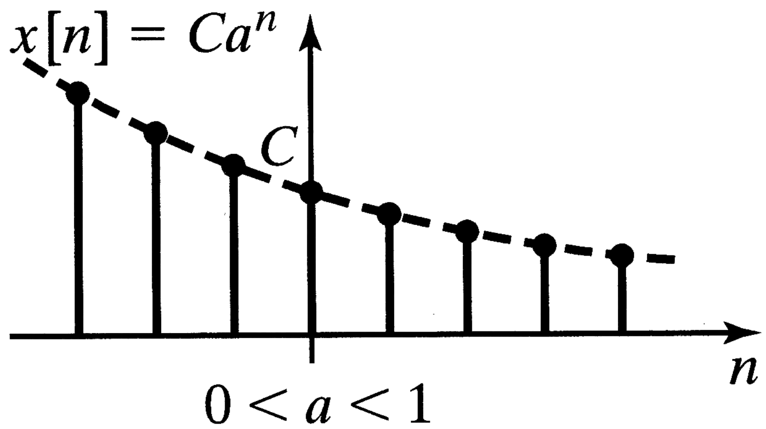
Suppose $x[0] = 1$

$$x[1] = ax[0] = a$$
$$x[2] = ax[1] = a^2$$
$$\vdots$$
$$x[n] = ax[n-1] = a^n$$

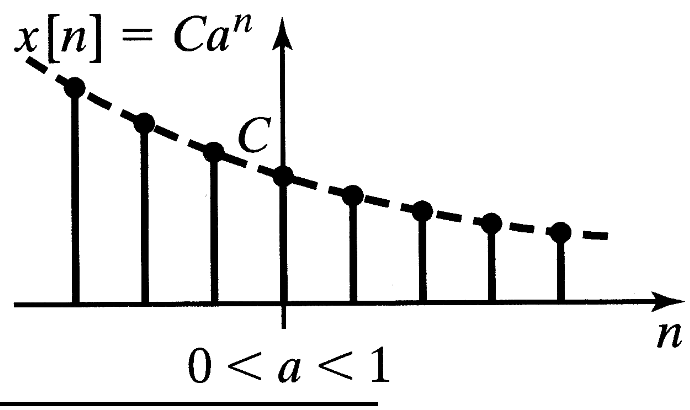
連續時間指數信號

- $e^{-bn} \rightarrow a^{-n}$
- Letting $a = e^{b}$, $b = \ln(a)$

$$x[n] = a^{n} = \left(e^{b}\right)^{n} = e^{bn}$$



Investigate the characteristics of the discrete-time signal $x[n] = a^n$



Let $a = e^b$

$$x[n] = a^n = \left(e^b\right)^n = e^{bn}$$

ex. $x[n] = 0.9^n$

$$0.9 = e^b \Rightarrow b = \ln 0.9 = -0.105$$

$$x[n] = 0.9^n = e^{-0.105n}$$

Suppose we sample an exponential signal every T seconds

$$x(t) = e^{-\sigma nT} = (e^{-\sigma T})^n = (a)^n, \text{ time constant } \tau = \frac{1}{\sigma}$$

$$\Rightarrow x[n] = \left(e^{-T/\tau}\right)^n = a^n$$

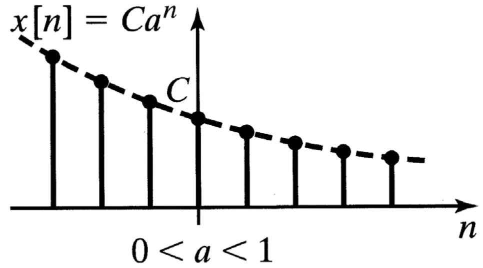
每個時間常數，取樣數目

$$e^{-T/\tau} = a \Rightarrow \frac{\tau}{T} = \frac{-1}{\ln a}$$

The number of samples per time constant τ / T

We can assign a time constant $\tau = \dfrac{-T}{\ln a}$

to the discrete exponential signal a^n



Ex. 9.7 For the signal $x[n = (0.8)^n$

$$\frac{\tau}{T} = \frac{-1}{\ln 0.8} = 4.48 \Rightarrow \tau = 4.48T$$

4.48 samples/time constant. Assume $nT > 4\tau$, amplitude can be neglected.

$$nT > 4\tau \approx 18T \Rightarrow n > 18$$

第18個以後的取樣點可以忽略不計

9.5 COMMON DISCRETE-TIME SIGNALS

■ System

A process for which cause-and-effect relations exist

Ex: the Euler integrator $y[n] = y[n-1] + Hx[n-1]$

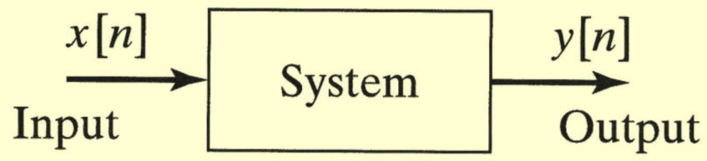


Figure 9.22 Block diagram for a discrete-time system.

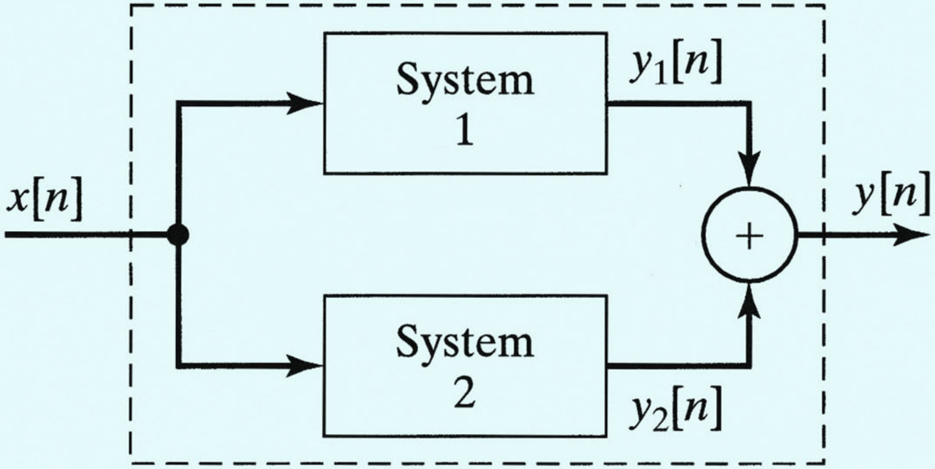
Example: A low-pass digital filter

the filter removes the higher frequencies in a signal, while passing the lower frequencies.

$$y[n] = T(x[n])$$
$$= (1-\alpha)y[n-1]$$ (an α-filter, 0<α<1)

Choices of α and the sample period T determine the range of frequencies that the filter will pass.

Interconnecting Systems



$$
\begin{aligned}
y[n] &= y_1[n] + y_2[n] \\
&= T_1[n] + T_2[n] \\
&= T(x[n])
\end{aligned}
$$

Figure 9.23 Parallel connection of systems.

$$
\begin{aligned}
y[n] &= T_2(y_1[n]) = T_2[T_1(x[n])] \\
&= T(x[n])
\end{aligned}
$$

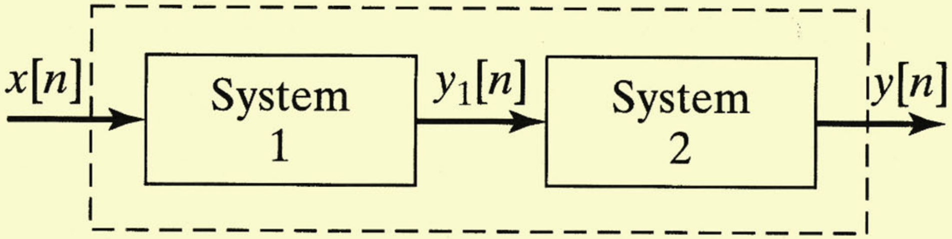


Figure 9.24 Series, or cascade, connection of systems.

9.6 PROPERTIES OF DISCRETE-TIME SYSTEMS

■ Systems with Memory

A system has memory if its output at time n_o, $y[no]$, depends on input values other than $x[n_o]$

Ex: A simple memoryless discrete-time system

$$y[n] = 5x[n]$$

(*a static system*)

Ex: An example of a system with memory

$$y[n] = y[n-1] + Hx[n-1]$$

(*a dynamic system*)

Invertibility

Distinct inputs results in distinct outputs

Ex: $y[n] = |x[n]|$

=> not invertible

Inverse of a System

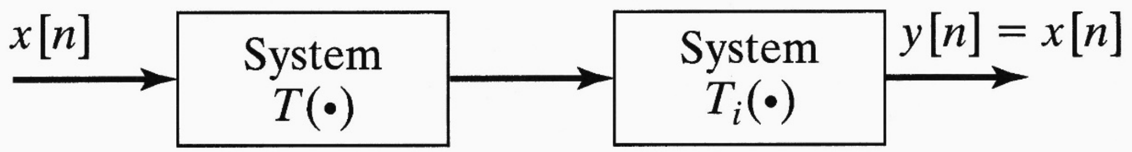


Figure 9.27 Identity system.

$$y[n] = T_i[T(x[n])] = x[n]$$

Causality

Causal Systems

A system is causal if the output at any time is dependent on the input only at the present time and in the past.

All physical systems are causal, whether continuous or discrete

Ex: unit delay $y[n] = x[n-1]$ causal

Ex: averaging system

$$y[n] = \frac{1}{3}[x[n-1] + x[n] + x[n+1]]$$ noncausal

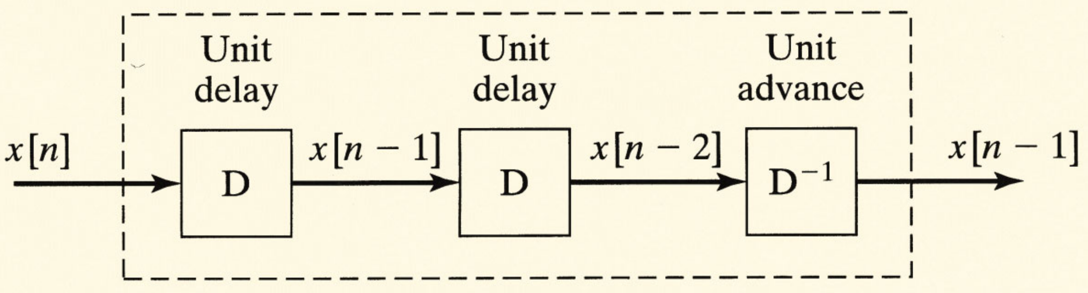


Figure 9.28 Realizable system with a unit advance.

■ Stability

BIBO Stability :

the output remains bounded for any bounded input

$$|x[n]| \le M \text{ for all } n.$$

$$\Rightarrow |y[n]| \le R \text{ for all } n.$$

Ex: the Euler integrator

$$y[n] = y[n-1] + Hx[n-1]$$

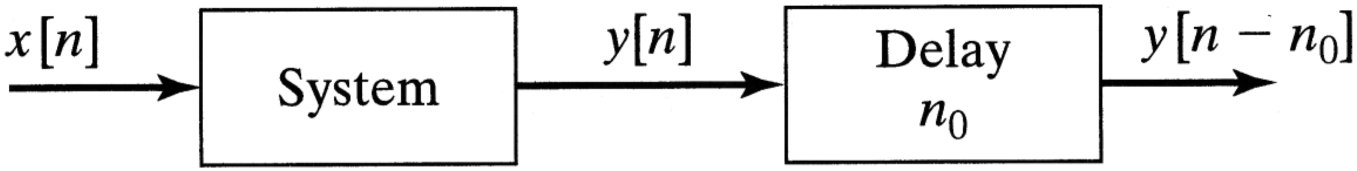
$$y[n] = H \sum_{k=-\infty}^{n-1} x[k]$$

is not stable when $x[k] = \text{constant}$

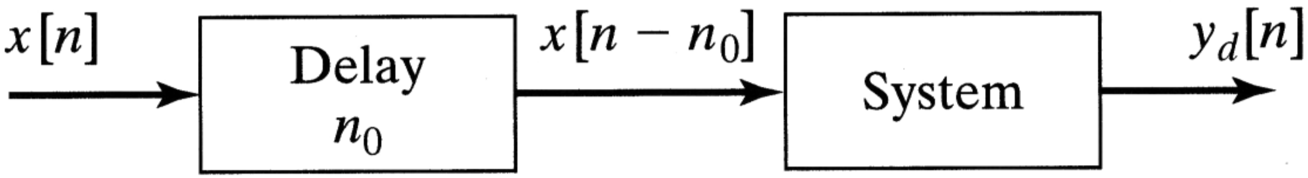
Time Invariance

if a time shift in the input results only in the same time shift in the output.

$$y[n]\Big|_{n-n_0} = y[n]\Big|_{x[n-n_0]}$$
$$x[n-n_0] \rightarrow y[n]|_{x[n-n_0]}$$
$$y[n-n_0] = y_d[n]$$



(a)



(b)

Linearity

$$a_1x_1[n] + a_2x_2[n] \rightarrow a_1y_1[n] + a_2y_2[n]$$

(superposition)

Ex1: Linear system

$$y[n] = Kx[n]$$

Ex2: Nonlinear system

$$y[n] = x^2[n]$$

Prove it by yourself!

Ex. 9.10 Illustrations of discrete-system properties

$$y[n] = \left[\frac{n+2.5}{n+1.5}\right]^2 x[n]$$

1. Memoryless
2. Invertible

$$x[n] = \left[\frac{n+1.5}{n+2.5}\right]^2 y[n]$$

3. Causal

   the output does not depend on the input at a future time

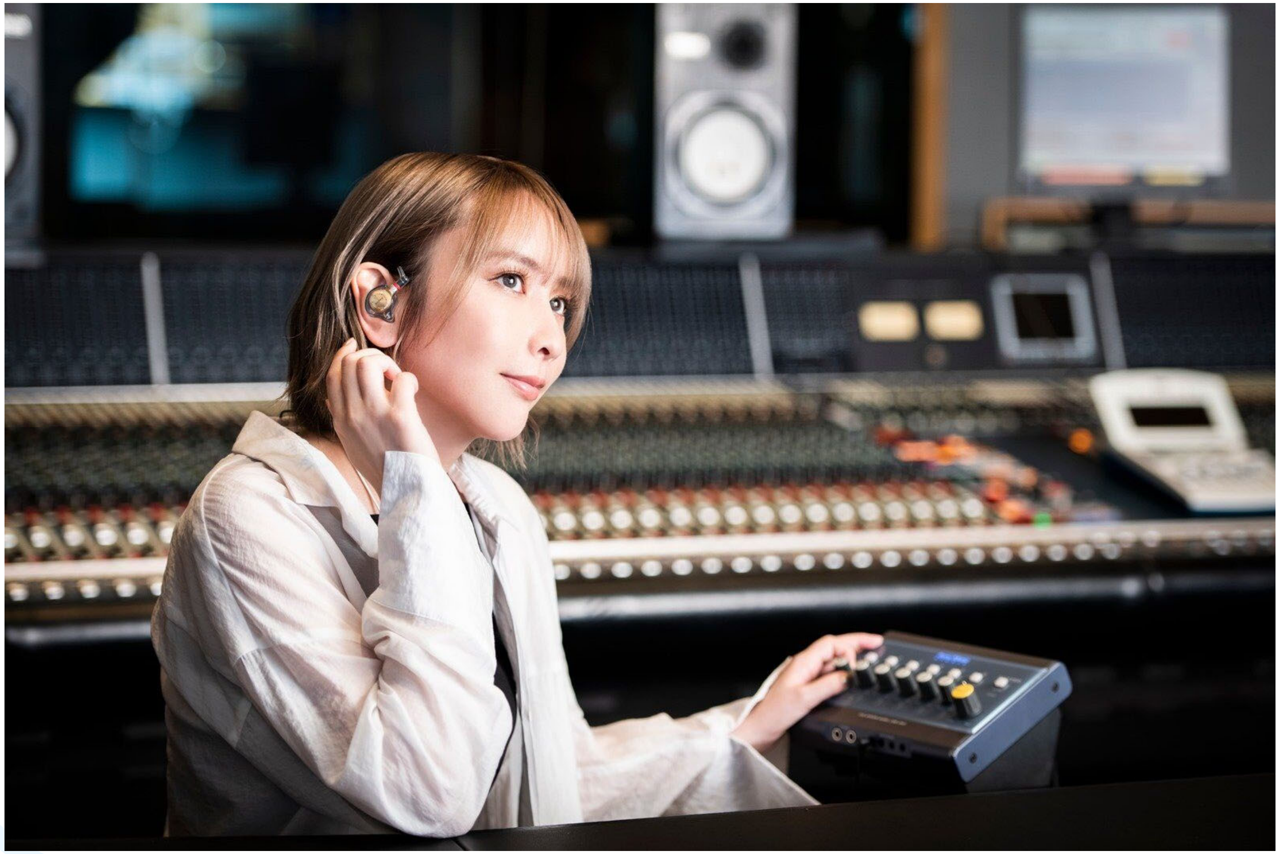
4. stable $|y[n]| \le 9M,\ \text{for}\ x[n] \le M$

   Prove it by yourself!

5. Not time-invariant $y[n]\big|_{n-n_0} \ne y[n]\big|_{x[n-n_0]}$

6. Linear

$$a_1x_1[n] + a_2x_2[n] \rightarrow \left[\frac{n+2.5}{n+1.5}\right]^2 a_1y_1[n] + a_2y_2[n]$$
$$= a_1y_1[n] + a_2y_2[n]$$



- The End of Chapter 9 -