CHAPTER III

3.1 (a) Since $F(+\infty) = 1$, we have $a = 1$.

And $p(x) = 1$, $x = 5$

$= 0$, elsewhere

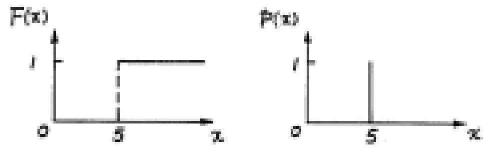

Figure 3.1a

(b) Since $F(+\infty) = 1$, we have $a = 1$

And $p(x) = 1/3$, $x = 5$

$= 2/3$, $x = 7$

$= 0$, elsewhere

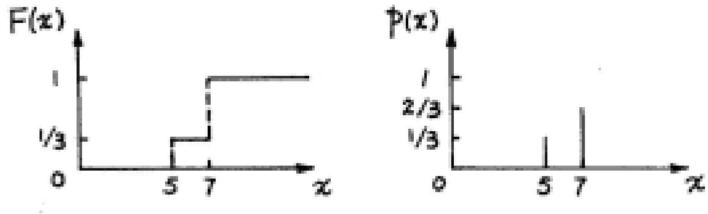

Figure 3.1b

(c) Since $F(x)$ is non-decreasing and $F(+\infty) = 1$,we have

$$F(+\infty) = \sum_{j=1}^{\infty} 1/a^j = \frac{1}{a-1} = 1 \text{ or } a = 2$$

And $p(x) = 1/2^x$, $x = 1, 2, \ldots$

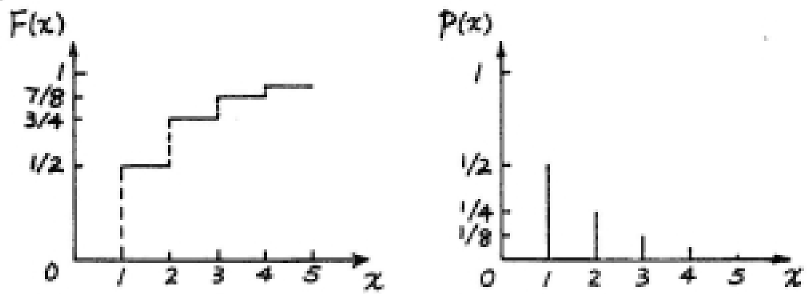

Figure 3.1c

(d) Since $F(x)$ is non-decreasing and $F(+\infty) = 1$, we have $a > 0$. And

$$f(x) = dF(x)/dx = ae^{-ax} \;,\; x > 0$$
$$= 0 \;, \quad \text{elsewhere}$$

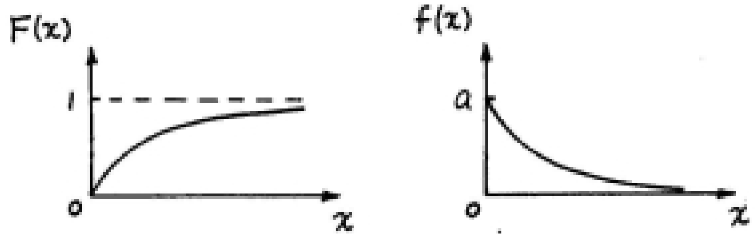

Figure 3.1d

(e) Since $F(x)$ is non-decreasing and $F(x) \leq 1$, we have $a \geq 0$. For $a > 0$,

$$f(x) = \frac{dF(x)}{dx} = ax^{a-1} \;,\; 0 \leq x \leq 1$$
$$= 0 \;, \quad \text{elsewhere}$$

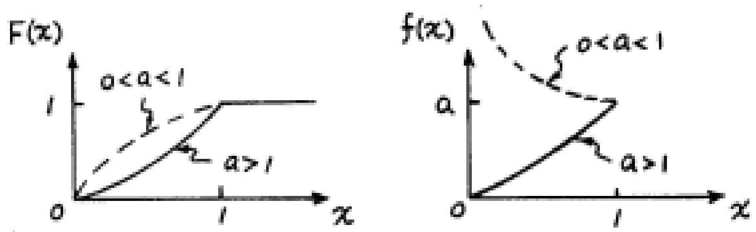

Figure 3.1e.1

For $a = 0$

$$p(x) = 1 \;,\; x = 0$$
$$= 0 \;,\; \text{elsewhere}$$

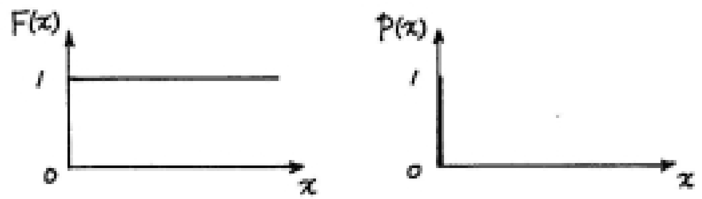

Figure 3.1e.2

(f) Assuming that X does not have a mixed distribution, we have

$$a \sin^{-1}\sqrt{1} = 1 \quad \text{or} \quad a = 2/\pi$$

$$f(x) = \frac{dF(x)}{dx} = \frac{1}{\pi\sqrt{x(1-x)}} \;,\; 0 \leq x \leq 1$$
$$= 0 \;, \quad \text{elsewhere}$$

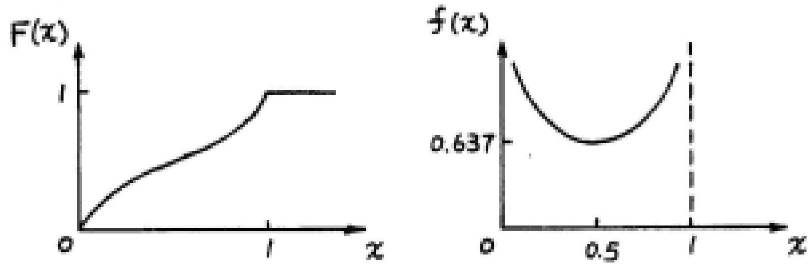

Figure 3.1f

(g) Since $F(+\infty) = 1$, we have $a + 1/2 = 1$ or $a = 1/2$. X has a mixed-type distribution, neither pmf nor pdf exists.

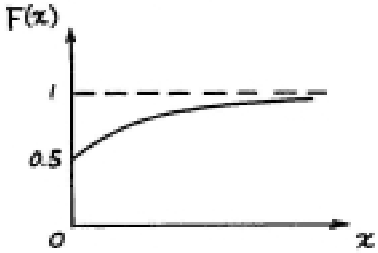

Figure 3.1g

3.2 (a) $P(x \le 6) = F(6) = 1$
$P(1/2 < x \le 7) = F(7) - F(1/2) = 1 - 0 = 1$

(b) $P(x \le 6) = F(6) = 1/3$
$P(1/2 < x \le 7) = F(7) - F(1/2) = 1 - 0 = 1$

(c) $P(x \le 6) = F(6) = \sum_{j=1}^{6} 1/2^j = 63/64$

$P(1/2 < x \le 7) = F(7) - F(1/2) = \sum_{j=1}^{7} 1/2^j - 0 = 127/128$

(d) $P(x \le 6) = F(6) = 1 - e^{-6a}$
$P(1/2 < x \le 7) = F(7) - F(1/2) = (1 - e^{-7a}) - (1 - e^{-a/2}) = e^{-a/2} - e^{-7a}$, $a > 0$

(e) $P(x \le 6) = F(6) = 1$
$P(1/2 < x \le 7) = F(7) - F(1/2) = 1 - (1/2)^a$, $a > 0$

(f) $P(x \le 6) = F(6) = 1$
$P(1/2 < x \le 7) = F(7) - F(1/2) = 1 - \frac{2}{\pi} \sin^{-1} \frac{1}{\sqrt{2}} = 1 - \frac{2}{\pi} \left(\frac{\pi}{4}\right) = \frac{1}{2}$

(g) $P(x \le 6) = F(6) = \frac{1}{2}(1 - e^{-1/3} + 1) = \frac{1}{2}(2 - e^{-1/3})$
$P(1/2 < x \le 7) = F(7) - F(1/2) = \frac{1}{2}(2 - e^{-7/2} - 2 + e^{-1/4}) = \frac{1}{2}(e^{-1/4} - e^{-7/2})$

3.3 (a)

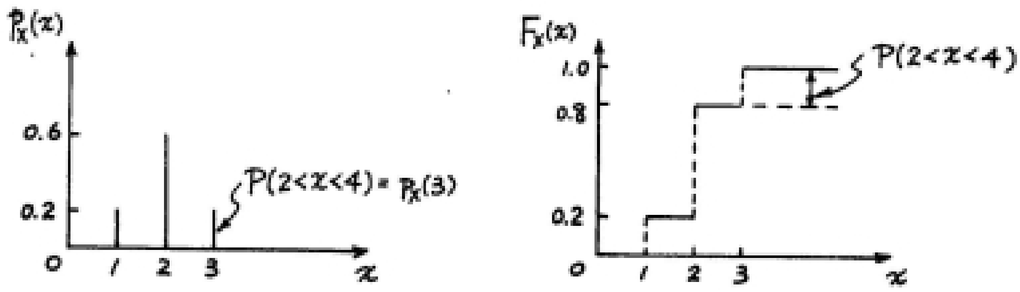

Figure 3.3a

(b)

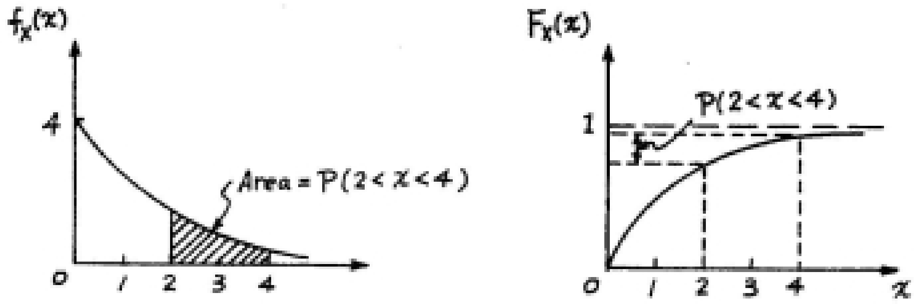

Figure 3.3b

3.4 $F_X(x) = \int_{-\infty}^{x} f_X(u)du$

(a) $F_X(x) = 0 \; , \; x < 90$

$= 0.1x - 9 \; , \; 90 \leq x < 100$

$= 1 \; , \quad x \geq 100$

(b) $F_X(x) = 0 \; , \; x < 0$

$= 2x - x^2 \; , \; 0 \leq x \leq 1$

$= 1 \; , \quad x > 1$

(c) $F_X(x) = \frac{1}{\pi} \tan^{-1} x + \frac{1}{2} \; , \; -\infty < x < \infty$

3.5 (a) $a = \frac{1}{3}$

(b)

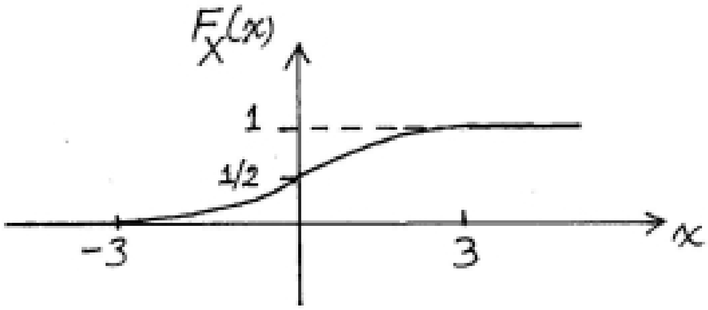

Figure 3.5b

(c) $P(X \geq 2) = \left(\frac{1}{9}\right)(1)\left(\frac{1}{2}\right) = \frac{1}{18}$

(d) $P(X \geq 2 | X \geq 1) = \dfrac{P(X \geq 2 \cap X \geq 1)}{P(X \geq 1)}$

$= \dfrac{P(X \geq 2)}{P(X \geq 1)} = \dfrac{1/18}{(2/9)(2)(1/2)} = \dfrac{1}{4}$

3.6 $P(X \geq 150) = \displaystyle\int_{150}^{\infty} \frac{100}{x^2} dx = \frac{100}{150} = \frac{2}{3}$

3.7 (a)

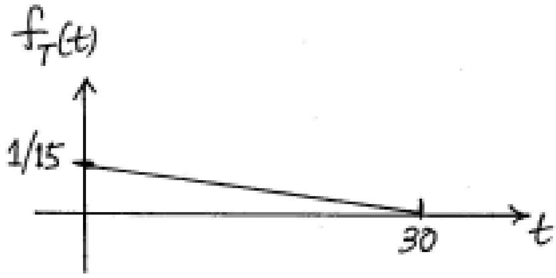

Figure 3.7a

(b) $F_T(t) = 0\ ,\qquad t < 0$

$$= \int_0^t f_T(u)du = \frac{t}{15} - \frac{t^2}{900}\ ,\quad 0 \le t \le 30$$

$$= 1\ ,\qquad t > 30$$

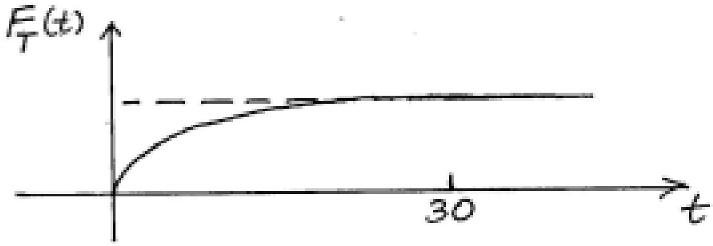

Figure 3.7b

(c) $P(T \ge 15) = \int_{15}^{\infty} f_T(t)dt$

$$= \int_{15}^{30} \left(\frac{1}{15} - \frac{t}{450}\right) dt = \frac{1}{30}(15)\left(\frac{1}{2}\right) = \frac{1}{4}$$

(d) $P(T \ge 15) = 1 - P(T \le 15)$

$$= 1 - F_T(15) = 1 - \left[\frac{15}{15} - \frac{15^2}{900}\right] = \frac{1}{4}$$

(e) $P(15 < T \le 16 | T \ge 15) = \dfrac{P(15 < T \le 16 \cap T \ge 15)}{P(T \ge 15)}$

$$= \frac{P(15 < T \le 16)}{P(T \ge 15)} = \int_{15}^{16} f_T(t)dt/(1/4)$$

$$= 0.322$$

3.8 Let X be the time of arrival in minutes. Then

Prob. desired = Shaded area/Total area

$$= \frac{\text{Length from 7:58 to 8:00}}{\text{Total length}}$$

$$= 2/5 = 0.4$$

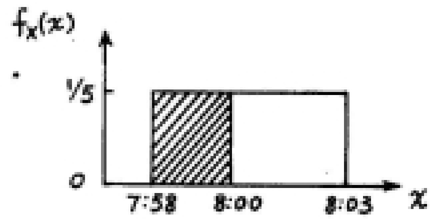

Figure 3.8

3.9 Consider only half of the bridge as shown in Figure. Then

$$F_X(x) = P(X \le x) = 0\ , \quad x < 0$$
$$= x/b\ , \quad 0 \le x \le b$$
$$= 1\ , \quad x > b$$

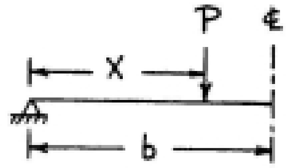

Figure 3.9.1

and

$$f_X(x) = \frac{dF_X(x)}{dx} = 1/b\ , \quad 0 \le x \le b$$
$$= 0, \quad \text{elsewhere}$$

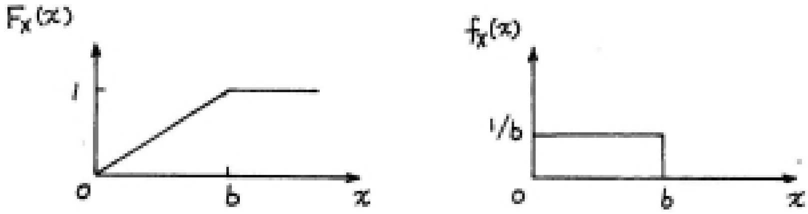

Figure 3.9.2

3.10 Consider only half the line segment AB and let Y be the distance from the midpoint of AB to the fire as shown in Figure. Then

$$F_X(x) = P(X \le x) = P(\sqrt{d^2 + Y^2} \le x)$$
$$= P(Y \le \sqrt{x^2 - d^2})$$
$$= F_Y(\sqrt{x^2 - d^2})$$

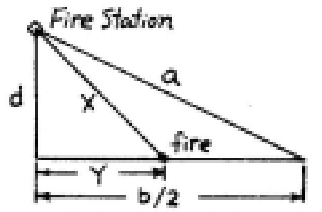

Figure 3.10.1

The r.v. Y is uniformly distributed over $(0, b/2)$ or

$$F_Y(y) = 0\ , \quad y < 0$$
$$= 2y/b\ , \quad 0 \le y \le b/2$$
$$= 1\ , \quad y > b/2$$

Then

$$F_X(x) = 0\ , \quad x < d$$
$$= \frac{2}{b}\sqrt{x^2 - d^2}\ , \quad d \le x \le a$$
$$= 1\ , \quad x > a$$

and

$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{2x}{b\sqrt{x^2 - d^2}}\ , \quad d \le x \le a$$
$$= 0\ , \quad \text{elsewhere}$$

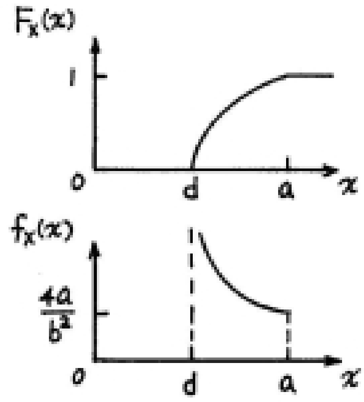

Figure 3.10.2

3.11 Let r_0 be the desired radius.

$$F_R(r_0) = \int_{-\infty}^{r_0} f_R(r)dr = \int_0^{r_0} ae^{-ar}dr$$
$$= 1 - e^{-ar_0} = 0.95$$
$$r_0 = -\frac{1}{a}\ln 0.05 = 3/a$$

3.12 Let Y be the r.v. representing the time instant at which the velocity v is applied. Then $f_Y(y)$ has the form

$$f_Y(y) = 1 \ , \ 0 \le y \le 1$$
$$= 0 \ , \ \text{elsewhere}$$

(a) It is clear that X is restricted to the range $(0, vt)$. Hence,

$$F_X(x) = 0 \ , \ x < 0$$
$$= 1 \ , \ x > vt$$

For $0 \le x \le vt$, we have
$X = v(t - Y)$ or $Y = t - \frac{X}{v}$

$$F_X(x) = P(X \le x)$$
$$= P[v(t-Y) \le x] = P(Y > t - \frac{x}{v})$$
$$= 1 - P(Y \le t - \frac{x}{v}) = 1 - F_Y(t - \frac{x}{v})$$

Since $F_Y(y) = y \ , 0 \le y \le 1$, we have

$$F_X(x) = 1 - t + \tfrac{x}{v} \ , \ 0 \le x \le vt$$

or

$$F_X(x) = 0 \ , \qquad x < 0$$
$$= 1 - t + \frac{x}{v} \ , \quad 0 \le x \le vt$$
$$= 1 \ , \qquad x > vt$$

(b) At $t = \frac{1}{2}$, $F_X(x) = \frac{1}{2} + \frac{x}{v}$, $0 \le x \le \frac{v}{2}$
$P(X \ge \frac{v}{3}) = 1 - F_X(\frac{v}{3}) = 1 - (\frac{1}{2} + \frac{1}{3}) = \frac{1}{6}$

3.13 (i) (a) $p_X(1) = p_{XY}(1,1) + p_{XY}(1,2) = 0.5 + 0.1 = 0.6$
$p_X(2) = p_{XY}(2,1) + p_{XY}(2,2) = 0.4$
$p_Y(1) = p_{XY}(1,1) + p_{XY}(2,1) = 0.6$
$p_Y(2) = p_{XY}(1,2) + p_{XY}(2,2) = 0.4$

(b) $p_{XY}(1,1) = 0.5$
$p_X(1)p_Y(1) = (0.6)(0.6) = 0.36$
Since $p_{XY}(1,1) \ne p_X(1)p_Y(1)$, they are not independent.

(ii) (a)
$$f_X(x) = \int_1^2 a(x+y)dy = a(x + \frac{3}{2}) \ , \ 0 \le x \le 1$$
$$= 0, \qquad \text{elsewhere}$$
$$f_Y(y) = \int_0^1 a(x+y)dx = a(y + \frac{1}{2}) \ , \ 1 \le y \le 2$$
$$= 0, \qquad \text{elsewhere}$$

(b) Since $f_{XY}(x,y) \neq f_X(x)f_Y(y)$, they are not independent.

(iii) (a) $f_X(x) = \int_0^\infty e^{-(x+y)} = e^{-x}\ ,\ x > 0$

$= 0\ ,$ elsewhere

$f_Y(y) = \int_0^\infty e^{-(x+y)} = e^{-y}\ ,\ y > 0$

$= 0\ ,$ elsewhere

(b) Since $f_{XY}(x,y) = f_X(x)f_Y(y)$, they are independent.

(iv) (a) $f_X(x) = \int_0^x 4y(x-y)e^{-(x+y)}dy = 4e^{-x}[e^{-x}(x+2)+(x-2)]\ ,\ x > 0$

$= 0\ ,$ elsewhere

$f_Y(y) = \int_y^\infty 4y(x-y)e^{-(x+y)}dx = 4ye^{-2y}\ ,\ y > 0$

$= 0\ ,$ elsewhere

(b) Since $f_{XY}(x,y) \neq f_X(x)f_Y(y)$, they are not independent.

3.14 (a) $p_X(x) = 0.1 + 0.2 = 0.3,\ x = 1$
$= 0.3 + 0.4 = 0.7,\ \ = 2$

$p_Y(y) = 0.1 + 0.3 = 0.4,\ y = 1$
$= 0.2 + 0.4 = 0.6,\ \ = 2$

(b) $P(X = 1) = p_X(1) = 0.3$

(c) $P(2X \leq Y) = p_{XY}(1,2) = 0.2$

3.15 Y_1, Y_2 and Y_3 also take values ± 1.

$$\begin{aligned} p_{Y_1}(1) &= P(X_1 = -1 \cap X_2 = -1) + P(X_1 = 1 \cap X_2 = 1) \\ &= p_{X_1}(-1)p_{X_2}(-1) + p_{X_1}(1)p_{X_2}(1) \\ &= 1/4 + 1/4 = 1/2 \end{aligned}$$

Similarly,

$p_{Y_1}(-1) = p_{X_1}(-1)p_{X_2}(1) + p_{X_1}(1)p_{X_2}(-1) = 1/2$

$p_{Y_2}(1) = 1/2\ ,\ p_{Y_2}(-1) = 1/2\ ,\ p_{Y_3}(1) = 1/2\ ,\ p_{Y_3}(-1) = 1/2$

Now

$$\begin{aligned} p_{Y_1Y_2}(1,1) &= P(X_1X_2 = 1 \cap X_2X_3 = 1) \\ &= P(X_1 = 1 \cap X_2 = 1 \cap X_3 = 1) + P(X_1 = -1 \cap X_2 = -1 \cap X_3 = -1) \\ &= p_{X_1}(1)p_{X_2}(1)p_{X_3}(1) + p_{X_1}(-1)p_{X_2}(-1)p_{X_3}(-1) \\ &= 1/8 + 1/8 = 1/4 \end{aligned}$$

Hence $p_{Y_1Y_2}(1,1) = p_{Y_1}(1)p_{Y_2}(1)$

Similarly, we can show that

$p_{Y_1Y_2}(1,-1) = p_{Y_1}(1)p_{Y_2}(-1)\ ,\ p_{Y_1Y_2}(-1,1) = p_{Y_1}(-1)p_{Y_2}(1)$

$p_{Y_1Y_2}(-1,-1) = p_{Y_1}(-1)p_{Y_2}(-1)$

Hence, Y_1 and Y_2 are independent.

Similar procedures show that Y_2 and Y_3 are independent and Y_1 and Y_3 are independent.

Consider

$$\begin{aligned} p_{Y_1Y_2Y_3}(1,1,1) &= P(X_1X_2 = 1 \cap X_1X_3 = 1 \cap X_2X_3 = 1) \\ &= P(X_1 = 1 \cap X_2 = 1 \cap X_3 = 1) + P(X_1 = -1 \cap X_2 = -1 \cap X_3 = -1) \\ &= p_{X_1}(1)p_{X_2}(1)p_{X_3}(1) + p_{X_1}(-1)p_{X_2}(-1)p_{X_3}(-1) \\ &= 1/4 \end{aligned}$$

But $p_{Y_1}(1)p_{Y_2}(1)p_{Y_3}(1) = (1/2)(1/2)(1/2) = 1/8$

Hence, $p_{Y_1Y_2Y_3}(1,1,1) \neq p_{Y_1}(1)p_{Y_2}(1)p_{Y_3}(1)$ and Y_1, Y_2 and Y_3 are not mutually independent.

3.16
$$\begin{aligned}F_{XY}(\infty,\infty) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy\\ &= \int_1^2\int_0^1 a(x+y)dxdy = \int_1^2 a\left(\frac{x^2}{2}+xy\right)\Big|_0^1 dy\\ &= \int_1^2 a\left(\frac{1}{2}+y\right)dy = a\left(\frac{y}{2}+\frac{y^2}{2}\right)\Big|_1^2\\ &= 2a = 1\end{aligned}$$

Or $a = 1/2$

(a) $P(X \leq 0.5 \cap Y > 1.0) = \int_1^2\int_0^{0.5} a(x+y)dxdy = 0.875a = 0.4375$

(b) $P(XY < 1/2) = \int_1^2\int_0^{\frac{1}{2y}} a(x+y)dxdy = 9/32$

(c)
$$\begin{aligned}P(X \leq 0.5|Y = 1.5) &= F_{XY}(0.5|1.5)\\ &= \int_{-\infty}^{x} f_{XY}(u|y)du \Big/ f_Y(y)\\ &= \left[\int_0^{0.5} a(x+y)dx \Big/ \int_0^1 a(x+y)dx\right]\Big|_{y=1.5}\\ &= 0.4375\end{aligned}$$

(d)
$$\begin{aligned}P(X \leq 0.5|Y \leq 1.5) &= P(X \leq 0.5 \cap Y \leq 1.5)/P(Y \leq 1.5)\\ &= \int_1^{1.5}\int_0^{0.5} a(x+y)dxdy \Big/ \int_1^{1.5}\int_0^1 a(x+y)dxdy\\ &= 0.429\end{aligned}$$

3.17
$$\begin{aligned}P(X \leq x|X \geq 100) &= \frac{P(X \leq x \cap X \geq 100)}{P(X \geq 100)} = \frac{P(100 \leq X \leq x)}{P(X \geq 100)}\\ &= \frac{F_X(x) - F_X(100)}{1 - F_X(100)} \quad , \quad x \geq 100\end{aligned}$$

3.18
$$P\left(X > b|X < \frac{b}{2}\right) = \frac{P(b < X < \frac{b}{2})}{P\left(X < \frac{b}{2}\right)} = \frac{\int_b^{b/2} 3x^2dx}{\int_{-1}^{b/2} 3x^2dx} = \frac{-7b^3}{b^3+8}$$

3.19 (a) $P(X > 3) = p_{XY}(4,1) + p_{XY}(5,0) = 0.0768 + 0.01024 = 0.087$

(b)
$$\begin{aligned}P(0 \leq Y < 3) &= p_{XY}(3,2) + p_{XY}(4,1) + p_{XY}(5.0)\\ &= 0.2304 + 0.0768 + 0.01024\\ &= 0.3174\end{aligned}$$

(c)
$$\begin{aligned}P(X > 3|Y \leq 2) &= \frac{P(X > 3 \cap Y \leq 2)}{P(Y \leq 2)}\\ &= \frac{p_{XY}(4,1) + p_{XY}(5,0)}{p_{XY}(3,2) + p_{XY}(4,1) + p_{XY}(5,0)}\\ &= \frac{0.087}{0.3174} = 0.274\end{aligned}$$

3.20 (a) $\int_0^2 \int_0^1 f_{XY}(x,y)dxdy$ gives

$$k = \frac{1}{(1-e^{-1})(1-e^{-2})}$$

(b)

$$f_X(x) = \int_0^2 f_{XY}(x,y)dy$$
$$= \frac{1}{(1-e^{-1})}e^{-x} \ , \ 0 < x < 1$$
$$= 0 \ , \qquad \text{elsewhere}$$

$$f_Y(y) = \int_0^1 f_{XY}(x,y)dx$$
$$= \frac{1}{(1-e^{-2})}e^{-y} \ , \ 0 < y < 2$$
$$= 0 \ , \qquad \text{elsewhere}$$

(c) Since $f_{XY}(x,y) = f_X(x)f_Y(y)$, X and Y are independent.

3.21 Let X be the driving time in minutes and Y be the time (in minutes) of leaving home after 7:30 a.m. Then $f_{XY}(x,y)$ has the form as shown in the figure.

$$P(\text{missing both trains}) = \text{Volume over shaded area}$$
$$= \frac{1}{300}(\frac{1}{2} \times 5 \times 5) = 0.042$$

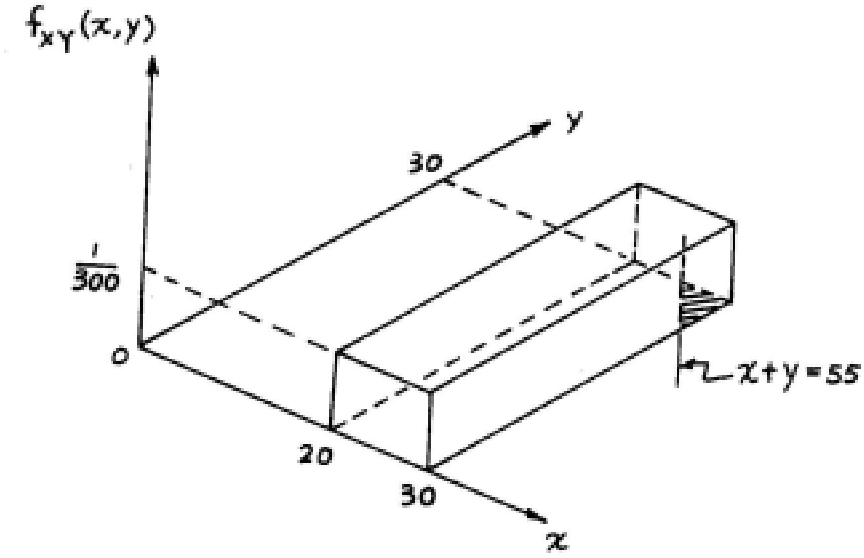

Figure 3.21

3.22
$$
\begin{aligned}
P(X \le 25 \cap Y > 8) &= \int_8^{\infty}\int_{-\infty}^{25} f_{XY}(x,y)dxdy \\
&= \int_8^{\infty} f_Y(y)dy \int_{-\infty}^{25} f_X(x)dx \\
&= \left[\frac{3}{64}\int_8^9 (9-y)^2 dy\right]\left[\frac{2}{2500}\int_0^{25} xdx\right] \\
&= 0.0039
\end{aligned}
$$

3.23 The jpdf of X and Y is

$$
\begin{aligned}
f_{XY}(x,y) &= 1 \ , \ 0 < (x,y) < 1 \\
&= 0 \ , \ \text{elsewhere}
\end{aligned}
$$

The required probability is the volume under $f_{XY}(x,y)$ with shaded area, shown below, as base. Hence,

$$
\begin{aligned}
P\left(XY < \frac{1}{2}\right) &= 1 - \int_{0.5}^{1}\int_{\frac{1}{2y}}^{1} f_{XY}(x,y)dxdy \\
&= 0.847
\end{aligned}
$$

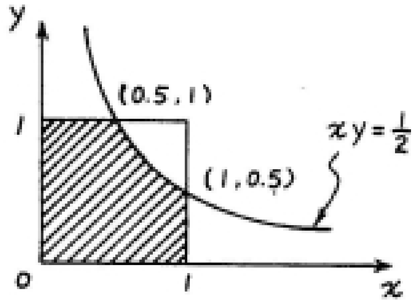

Figure 3.23

3.24 The required probability is the volume under $f_X(x)f_Y(y)$ over the base as shown in the Figure. Hence,

$$
P(X^2+Y^2 \le a^2) = \iint\limits_{x^2+y^2\le a^2} f_X(x)f_Y(y)dxdy
$$

Using polar coordinates ($x^2+y^2=r^2,\ dxdy = rdrd\theta$)

$$
\begin{aligned}
P(X^2+Y^2 \le a^2) &= \frac{1}{2\pi\sigma^2}\int_0^{2\pi}\int_0^{a} e^{-r^2/2\sigma^2} rdrd\theta \\
&= 1 - e^{-a^2/2\sigma^2}
\end{aligned}
$$

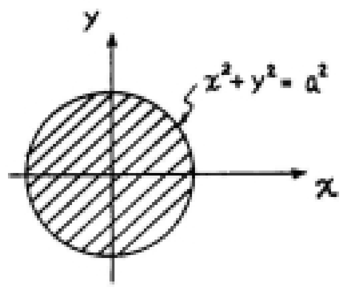

Figure 3.24

3.25
$$\begin{aligned} P[\min(X_1, X_2, \ldots, X_n) \le u] &= P(X_1 \le u \cup X_2 \le u \cup \ldots \cup X_n \le u) \\ &= 1 - P(X_1 > u \cap X_2 > u \cap \ldots \cap X_n > u) \\ &= 1 - P(X_1 > u)P(X_2 > u) \ldots P(X_n > u) \\ &= 1 - [1 - F_X(u)]^n \end{aligned}$$

$$\begin{aligned} P[\max(X_1, X_2, \ldots, X_n) \le u] &= P(X_1 \le u \cap X_2 \le u \cap \ldots \cap X_n \le u) \\ &= P(X_1 \le u)P(X_2 \le u) \ldots P(X_n \le u) \\ &= [F_X(u)]^n \end{aligned}$$

3.26 Let

$$p_j = \begin{bmatrix} p_{X_j}(1) \\ p_{X_j}(2) \\ \vdots \\ p_{X_j}(7) \end{bmatrix} \quad \text{and } P = [P_{ij}] = [P(X_{k+1} = i | X_k = j)]$$

(a)

$$p_3 = P^3 p_0 = P^3 \begin{bmatrix} 0.00 \\ 0.00 \\ 0.04 \\ 0.06 \\ 0.11 \\ 0.28 \\ 0.51 \end{bmatrix} = \begin{bmatrix} 0.016 \\ 0.035 \\ 0.080 \\ 0.125 \\ 0.415 \\ 0.192 \\ 0.137 \end{bmatrix}$$

(b) $p_{X_4X_3}(i,j) = p_{X_4X_3}(i|j)p_{X_3}(j)$. Hence,

$$P_{X_4X_3}(1,1) = p_{X_4X_3}(1|1)p_{X_3}(1) = (0.388)(0.016) = 0.006$$

and the others follow in a similar fashion. The result is

Table of $p_{X_4X_3}(i,j)$

	$j=1$	2	3	4	5	6	7
$i=1$	0.006	0.004	0.003	0.003	0.004	0.000	0.000
2	0.002	0.009	0.008	0.005	0.010	0.002	0.001
3	0.003	0.008	0.015	0.014	0.031	0.008	0.005
4	0.001	0.004	0.015	0.027	0.051	0.017	0.011
5	0.002	0.007	0.029	0.054	0.196	0.075	0.050
6	0.001	0.002	0.005	0.015	0.071	0.060	0.032
7	0.000	0.001	0.005	0.008	0.052	0.030	0.038