CHAPTER IV

4.1 All integrals can be carried out by using either direct integration, integration by parts, table look-up or computer software.

(a)
$$p_X(x) = 1 \ , \ x = 5$$
$$= 0 \ , \ \text{elsewhere}$$
$$m_X = \sum_i x_i p_X(x_i) = 5p_X(5) = 5$$
$$\sigma_X^2 = \sum_i (x_i - m_X)^2 p_X(x_i) = (5-5)^2 p_X(5) = 0$$

(b)
$$p_X(x) = 1/3 \ , \ x = 5$$
$$= 2/3 \ , \ x = 7$$
$$= 0 \ , \quad \text{elsewhere}$$
$$m_X = 5(1/3) + 7(2/3) = 19/3$$
$$\sigma_X^2 = (5 - 19/3)^2(1/3) + (7 - 19/3)^2(2/3) = 8/9$$

(c)
$$p_X(x) = 1/2^x \ , \ x = 1, 2, \ldots$$
$$m_X = \sum_{k=1}^{\infty} k(1/2^k) = 2$$
$$\alpha_2 = E\{X^2\} = \sum_{k=1}^{\infty} k^2(1/2^k) = 6$$
$$\sigma_X^2 = \alpha_2 - m_X^2 = 6 - 4 = 2$$

(d)
$$f_X(x) = ae^{-ax} \ , \ x > 0$$
$$= 0 \ , \qquad \text{elsewhere}$$
$$m_X = \int_0^{\infty} x[ae^{-ax}]dx = 1/a$$
$$\alpha_2 = E\{X^2\} = \int_0^{\infty} x^2[ae^{-ax}]dx = 2/a^2$$
$$\sigma_X^2 = 2/a^2 - 1/a^2 = 1/a^2$$

(e) Consider the case $a > 0$.

$$f_X(x) = ax^{a-1}\ ,\ 0 \le x \le 1$$
$$= 0\ ,\quad \text{elsewhere}$$
$$m_X = \int_0^1 x(ax^{a-1})dx = \frac{a}{a+1}$$
$$\sigma_X^2 = \int_0^1 \left(x - \frac{a}{a+1}\right)^2 (ax^{a-1})dx = \frac{a}{(a+2)(a+1)^2}$$

(f) $$f_X(x) = \frac{1}{\pi\sqrt{x(1-x)}}\ ,\ 0 \le x \le 1$$
$$= 0\ ,\quad \text{elsewhere}$$
$$m_X = \frac{1}{\pi}\int_0^1 \frac{x}{\sqrt{x(1-x)}}dx = \frac{1}{2}$$
$$\alpha_2 = E\{X^2\} = \frac{1}{\pi}\int_0^1 \frac{x^2}{\sqrt{x(1-x)}}dx = \frac{3}{8}$$
$$\sigma_X^2 = \alpha_2 - m_X^2 = \frac{3}{8} - \frac{1}{4} = \frac{1}{8}$$

(g) $$f_X(x) = \frac{1}{4}e^{-x/2}\ ,\ x > 0$$
$$p_X(x) = \frac{1}{2}\ ,\quad x = 0$$
$$f_X(x) = 0\ ,\quad \text{elsewhere}$$

Following the mass analogy, we have
$$m_X = (0)p_X(0) + \int_0^\infty x\left(\frac{1}{4}e^{-x/2}\right)dx = 1$$
$$\alpha_2 = E\{X^2\} = \int_0^\infty x^2\left(\frac{1}{4}e^{-x/2}\right)dx = 4$$
$$\sigma_X^2 = \alpha_2 - m_X^2 = 3$$

4.2 (a) $$m_X = \int_{90}^{100} x(0.1)dx = 95$$
$$\sigma_X^2 = \int_{90}^{100} (x-95)^2(0.1)dx = \frac{25}{3}$$

(b) $$m_X = \int_0^1 x[2(1-x)]dx = \frac{1}{3}$$
$$\sigma_X^2 = \int_0^1 (x - \frac{1}{3})^2[2(1-x)]dx = \frac{1}{18}$$

(c) $$m_X = \int_{-\infty}^{\infty} x\left[\frac{1}{\pi(1+x^2)}\right]dx$$

Consider
$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left|\frac{x}{1+x^2}\right|dx = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2}dx = \lim_{u\to\infty}\frac{2}{\pi}\ln(1+u^2) = \infty$$

Hence, m_X does not exist. Similar procedure shows that the variance, σ_X^2, as well as all higher moments, does not exist.

4.3
$$f_X(x) = \frac{1}{3}e^{-x/3}\ ,\ 0 \le x < 3$$
$$= \frac{1}{6}e^{-x/3}\ ,\ x > 3$$
$$p_X(x) = \frac{1}{2}e^{-1}\ ,\ x = 3$$
$$f_X(x) = 0\ ,\quad \text{elsewhere}$$

As in Prob. 4.1(g), we have
$$m_X = 3\left(\frac{1}{2}e^{-1}\right) + \int_0^3 x\left(\frac{1}{3}e^{-x/3}\right)dx + \int_3^\infty x\left(\frac{1}{6}e^{-x/3}\right)dx$$
$$= 2.44\,\text{min}$$

4.4 Let $a = \dfrac{2\sigma_R^3}{0.9996\pi}$, $b = 1 + \sigma_R^2$, $c = 0.33$. We have
$$f_R(r) = \frac{a}{(r^2 - 2r + b)^2}\ ,\ r \ge c$$
$$= 0\ ,\qquad \text{elsewhere}$$
$$m_R = \int_c^\infty \frac{ar}{(r^2 - 2r + b)^2}dr = \frac{a}{2}\left[\frac{b-c}{(1-b)(c^2-2c+b)} + \frac{1}{2(1-b)^{3/2}}\ln\frac{c-1-\sqrt{1-b}}{c-1+\sqrt{1-b}}\right]$$

4.5 Let X be the score.
$$E\{X\} = 4P(0 \le X < 1/\sqrt{3}) + 3P(1/\sqrt{3} \le X < 1) + 2(1 \le X < \sqrt{3})$$
$$P(0 \le X < 1/\sqrt{3}) = \int_0^{1/\sqrt{3}} \frac{2}{\pi(1+r^2)}dr = 1/3$$
Similarly,
$$P(1/\sqrt{3} \le X < 1) = \frac{1}{6}$$
$$P(1 \le X < \sqrt{3}) = \frac{1}{6}$$
Hence,
$$E\{X\} = 4\left(\frac{1}{3}\right) + 3\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) = \frac{13}{6}$$

4.6 (a) $\displaystyle\int_0^\infty ae^{-x/2}dx = 1$ gives $a = \dfrac{1}{2}$

(b) $m_X = 2$, $\sigma_X^2 = 4$. [see Prob. 4.1(d)].

(c) $\displaystyle m_Y = E\left\{\frac{X}{2} - 1\right\} = \int_0^\infty \left(\frac{x}{2} - 1\right)\frac{1}{2}e^{-x/2}dx = 0$
$$\sigma_Y^2 = E\left\{\left(\frac{X}{2} - 1\right)^2\right\} = \int_0^\infty \left(\frac{x}{2} - 1\right)^2\frac{1}{2}e^{-x/2}dx = 1$$

4.7 $m_Y = am + b$
$\sigma_Y^2 = a^2\sigma^2$
Setting $m_Y = 0$ and $\sigma_Y^2 = 1$, we have $a = \frac{1}{\sigma}$ and $b = -\frac{m}{\sigma}$

4.8 Let T_1 and T_2 be, respectively, morning and evening waiting times in minutes. Then

$$m_{T_1} = 2.5 \;, \quad \sigma_{T_1}^2 = \frac{25}{12}$$
$$m_{T_2} = \frac{10}{3} \;, \quad \sigma_{T_2}^2 = \frac{50}{9}$$

(a) mean $= 5m_{T_1} + 5m_{T_2} = 29.17\,\text{min}$
(b) variance $= 5\sigma_{T_1}^2 + 5\sigma_{T_2}^2 = 34.03\,\text{min}^2$
(c) mean $= m_{T_1} - m_{T_2} = 0.42\,\text{min}$
variance $= \sigma_{T_1}^2 + \sigma_{T_2}^2 = 6.81\,\text{min}^2$
(d) mean $= 5m_{T_1} - 5m_{T_2} = 2.10\,\text{min}$
variance $= 5(\sigma_{T_1}^2 + \sigma_{T_2}^2) = 34.03\,\text{min}^2$

4.9 (a) $E\{X\} = \int_0^1 x(6x)(1-x)dx = \frac{1}{2}$

(b) $E\left\{\frac{\pi}{4}X^2\right\} = \int_0^1 \left(\frac{\pi}{4}x^2\right) 6x(1-x)dx = \frac{3\pi}{40}$

4.10 It is given that $\int_a^b f_X(x)dx = 1$.

(a)
$$m_X = \int_a^b x f_X(x)dx \le b\int_a^b f_X(x)dx = b$$
$$\ge a\int_a^b f_X(x)dx = a$$

Hence, $a \le m_X \le b$.

(b) Let
$$\int_a^{m_X} f_X(x)dx = p \text{ and } \int_{m_X}^b f_X(x)dx = 1 - p$$
$$\sigma_X^2 = \int_a^{m_X}(x-m_X)^2 f_X(x)dx + \int_{m_X}^b (x-m_X)^2 f_X(x)dx$$
$$\le (a - m_X^2)p + (b - m_X)^2(1-p) \text{ for all } m_X \text{ and } p.$$

The minimum of the right-hand side occurs at $m_X = \frac{a+b}{2}$ and $p = \frac{1}{2}$, which can be verified by taking partial derivatives of the right-hand side with respect to m_X and p and setting them to zero. Hence,

$$\sigma_X^2 \le \left(a - \frac{a+b}{2}\right)^2\left(\frac{1}{2}\right) + \left(b - \frac{a+b}{2}\right)^2\left(\frac{1}{2}\right) = \frac{(b-a)^2}{4}$$

4.11 Chebyshev Inequality (4.17) gives

$$P(|X - m_X| \ge k) \le \sigma_X^2/k^2$$

or

$P(|X - m_X| < k) \geq 1 - \sigma_X^2/k^2$

If $\sigma_X^2 = 0$, then $P(|X - m_X| < k) = 1$ for arbitrarily small k. Thus

$$P(X = m_X) = 1$$

4.12 $f_T(t) = (1-p)\lambda e^{-\lambda t}$, $t > 0$
$p_T(t) = p$, $t = 0$
$f_T(t) = 0$, elsewhere

(a) As in Prob. 4.1(g), we have

$$m_T = 0(p) + \int_0^\infty t(1-p)\lambda e^{-\lambda t} dt = \frac{1-p}{\lambda}$$

(b) (See Example 3.9)

$$f_T(t|T > 0) = \frac{(1-p)\lambda e^{-\lambda t}}{1-p} = \lambda e^{-\lambda t} \ , \ t > 0$$

$$E\{T|T > 0\} = \int_0^\infty t f_Y(t|T > 0) dt = \frac{1}{\lambda}$$

4.13 Let X = driving time in minutes
Y = time (in minutes) of leaving home after 7:30 a.m.

Then

$$f_{XY}(x,y) = \frac{1}{300} \ , \quad 20 \leq x \leq 30 \ , \ 0 \leq y \leq 30$$
$$= 0 \ , \quad \text{elsewhere}$$

$$\begin{aligned} \text{Prob. of catching first train} &= P(X + Y \leq 35) \\ &= \int_{20}^{30} \int_0^{35-x} f_{XY}(x,y) dy dx \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \text{Prob. of catching second train} &= P(X + Y \leq 55) - P(X + Y \leq 35) \\ &= \int_{20}^{30} \int_0^{55-x} f_{XY}(x,y) dy dx - \frac{1}{3} \\ &= 0.62 \end{aligned}$$

Arrival time of first train = 8:35 a.m. = 8 hr + 35 min
Arrival time of second train = 9:00 a.m. = 8 hr + 60 min

$$\begin{aligned} E\{\text{Arrival time} \mid \text{catching one train}\} &= 8 \text{ hr } + \frac{35(1/3) + 60(0.62)}{1/3 + 0.62} \\ &= 8 \text{ hr } + 51 \text{ min} \\ &= 8:51 \text{ a.m.} \end{aligned}$$

4.14 (See Example 4.8)

Let $Y = 1, 2$ be the events that the miner chooses to go to the right and to the left, respectively. Then $P(Y = 1) = 1/2$ and $P(Y = 2) = 1/2$. Let X be the time to safety in minutes.

$$E\{X\} = E\{X|Y = 1\}P(Y = 1) + E\{X|Y = 2\}P\{Y = 2\}$$
$$= [3 + E\{X\}]\left(\frac{1}{2}\right) + \left[5\left(\frac{1}{3}\right) + (5 + E\{X\})\left(\frac{2}{3}\right)\right]\left(\frac{1}{2}\right) = 4 + \frac{5}{6}E\{X\}$$
$$E\{X\} = 24\,\text{min}$$

4.15 Assume that Y is continuous

(a) $E\{X|Y = y\} = \int_{-\infty}^{\infty} x f_{XY}(x|y)dx$

If X and Y are independent, $f_{XY}(x|y) = f_X(x)$. Hence,

$$E\{X|Y = y\} = \int_{-\infty}^{\infty} X f_X(x)dx = E\{X\}$$

(b) $E\{XY|Y = y\} = E\{Xy|Y = y\} = yE\{X|Y = y\}$

(c) From Eq. (4.13), $E\{XY\} = E\{E\{XY|Y\}\}$
From (b), $E\{XY|Y\} = YE\{X|Y\}$
Hence, $E\{XY\} = E\{YE\{X|Y\}\}$

4.16 From Chebyshev Inequality, we have

$P(|X - 1| \leq 0.75) \geq 1 - \sigma_X^2/0.75^2$

$\sigma_X^2 = \int_0^2 (x-1)^2\left(\frac{1}{2}\right)dx = \frac{1}{3}$

Hence,

$P(|X - 1| \leq 0.75) \geq 1 - 0.59 = 0.41$

The exact probability is given by

$P(|X - 1| \leq 0.75) = \text{Shaded area } = \left(\frac{1}{2}\right)(1.5) = 0.75$

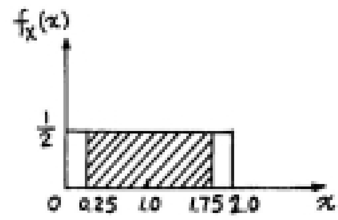

Figure 4.16

4.17 From Chebyshev Inequality,

$$P(|X - 1| \leq h) \geq 1 - \frac{1}{3h^2}\ , \quad \frac{1}{3h^2} < 1$$
$$\geq 0\ , \quad \frac{1}{3h^2} \geq 1$$

From uniform distribution

$P(|X - 1| \leq h) = \left(\frac{1}{2}\right)(2h) = h$

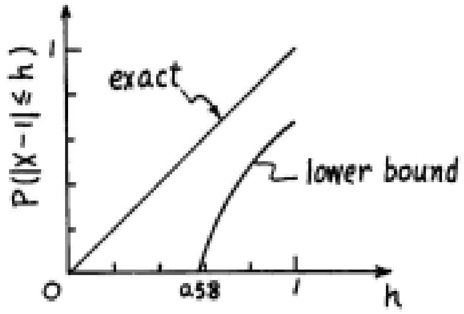

Figure 4.17

4.18 $m_X = \int_0^{\infty} x f_X(x)dx$

$$= \int_0^a x f_X(x)dx + \int_a^{\infty} x f_X(x)dx$$
$$\geq \int_a^{\infty} x f_X(x)dx$$
$$\geq \int_a^{\infty} a f_X(x)dx = a\int_a^{\infty} f_X(x)dx = aP(X \geq a)$$

4.19 (a) Using Markov's Inequality (Prob. 4.18)

$P(X \geq 85) \leq 70/85$ or $P(X < 85) \geq 15/85$

$P(X \geq 55) \leq 70/55$ or $P(X \geq 55) \leq 1$

Hence, we can only say that

$P(55 \leq X < 85) \geq 0$

(b) Using Chebyshev Inequality [Eq. (4.17)],

$P(|X-70| \geq 15) \leq \sigma_X^2/15^2 = (10/15)^2 = 4/9$

Hence,

$P(55 \leq X < 85) \geq 1 - 4/9 = 5/9$, a much more improved bound.

4.20 $m_X = \sum_{k=0}^{\infty} k p_X(k) = 100$

$$E\{X^2\} = \sum_{k=0}^{\infty} k^2 p_X(k) = 100^2 + 100$$

$$\sigma_X^2 = E\{X^2\} - m_X^2 = 100$$

$$P(80 \leq X \leq 120) = P(|X-100| \leq 20) \geq 1 - \frac{\sigma_X^2}{20^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

Hence, a lower bound is 3/4.

4.21 (i) $m_X = \sum_{i,j} x_i p_{XY}(x_i, y_j) = (1)(0.5) + (1)(0.1) + (2)(0.1) + (2)(0.3)) = 1.4$

$$m_Y = \sum_{i,j} y_j p_{XY}(x_i, y_j) = 1.4$$

$$\alpha_{20} = \sum_{i,j} x_i^2 p_{XY}(x_i, y_j) = 2.2$$

$$\alpha_{02} = \sum_{i,j} y_j^2 p_{XY}(x_i, y_j) = 2.2$$

$$\alpha_{011} = \sum_{i,j} x_i y_j p_{XY}(x_i, y_j) = 2.1$$

$$\sigma_X^2 = \alpha_{20} - m_X^2 = 0.24$$

$$\sigma_Y^2 = \alpha_{02} - m_Y^2 = 0.24$$

$$\rho = \frac{\mu_{11}}{\sigma_X \sigma_Y} = \frac{\alpha_{11} - m_X m_Y}{\sigma_X \sigma_Y} = 0.58$$

(ii) $m_X = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{XY}(x,y) dx dy = a \int_1^2 \int_0^1 x(x+y) dx dy = \frac{13a}{12}$

$$m_Y = a \int_1^2 \int_0^1 y(x+y) dx dy = \frac{37}{12} a$$

$$\alpha_{20} = a \int_1^2 \int_0^1 x^2(x+y) dx dy = \frac{3}{4} a$$

$$\alpha_{02} = a \int_1^2 \int_0^1 y^2(x+y) dx dy = \frac{59}{12} a$$

$$\alpha_{11} = a \int_1^2 \int_0^1 xy(x+y) dx dy = \frac{5}{3} a$$

$$\sigma_X^2 = \alpha_{20} - m_X^2 = \frac{3}{4} a - \left(\frac{13}{12} a\right)^2$$

$$\sigma_Y^2 = \alpha_{02} - m_Y^2 = \frac{59}{12}a - \left(\frac{37}{12}a\right)^2$$

$$\rho = \frac{\alpha_{11} - m_X m_Y}{\sigma_X \sigma_Y}$$

(iii) X and Y are independent with

$$f_X(x) = e^{-x}\ ,\ x > 0 \qquad f_Y(y) = e^{-y}\ ,\ y > 0$$
$$= 0\ ,\quad \text{elsewhere} \qquad = 0\ ,\quad \text{elsewhere}$$

From the results of Prob. 4.1(d),

$m_X = m_Y = 1$
$\sigma_X^2 = \sigma_Y^2 = 1$
$\rho = 0$

(iv) $m_X = \int_0^\infty \int_0^x x[4y(x-y)e^{-(x+y)}]dydx = 3$

Similarly, we have

$m_Y = 1\ ,\quad \sigma_X^2 = 2.5\ ,\quad \sigma_Y^2 = 0.5\ ,\quad \rho = 0.45$

4.22 $m_R = 1000$ ohms

$$\sigma_R^2 = \frac{(1100-900)^2}{12} = \frac{10000}{3}\ \text{ohms}^2$$

(a) $m_V = E\{(R + r_0)i\} = (m_R + r_0)i = 20$ volts

(b) $\sigma_V^2 = i^2\sigma_R^2 = \frac{1}{3}$ volts^2

4.23
$$E\{Z\} = E\{\sqrt{X^2+Y^2}\} = \int_0^2 \int_0^1 \sqrt{x^2+y^2}(xy)dxdy$$
$$= \int_0^2 (y/3)[(1+y^2)^{3/2} - y^3]dy$$
$$= 1.53$$

4.24
$$m_Z = E\{XY\} = E\{X\}E\{Y\} = m_X m_Y$$
$$\alpha_2 = E\{Z^2\} = E\{X^2\}E\{Y^2\} = (\sigma_X^2 + m_X^2)(\sigma_Y^2 + m_Y^2)$$
$$\sigma_Z^2 = \alpha_2 - m_Z^2 = (\sigma_X^2 + m_X^2)(\sigma_Y^2 + m_Y^2) - m_X^2 m_Y^2$$
$$= \sigma_X^2\sigma_Y^2 + m_X^2\sigma_Y^2 + m_Y^2\sigma_X^2$$

4.25
$$\mu_{11} = E\{(X - m_X)(Y - m_Y)\} = E\{[(X_1 - m_{X_1}) + (X_2 - m_{X_2})][(X_2 - m_{X_2}) + (X_3 - m_{X_3})]\}$$
$$= E\{(X_1 - m_{X_1})(X_2 - m_{X_2}) + (X_1 - m_{X_1})(X_3 - m_{X_3})$$
$$+ (X_2 - m_{X_2})^2 + (X_2 - m_{X_2})(X_3 - m_{X_3})\}$$
$$= 0 + 0 + \sigma_{X_2}^2 + 0 = \sigma_{X_2}^2$$

$\sigma_X^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2\ ,\ \sigma_Y^2 = \sigma_{X_2}^2 + \sigma_{X_3}^2$

Hence,

$$\rho_{XY} = \frac{\mu_{11}}{\sigma_X\sigma_Y} = \frac{\sigma_{X_2}^2}{\sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2}\sqrt{\sigma_{X_2}^2 + \sigma_{X_3}^2}}$$

4.26
$$\begin{aligned} p_X(i) &= 2a+b\ , \quad i=-1 & p_Y(j) &= 2a+b\ , \quad j=-1 \\ &= 2b\ , \quad\quad\ \ =0 & &= 2b\ , \quad\quad\ \ =0 \\ &= 2a+b\ , \quad\ =1 & &= 2a+b\ , \quad\ =1 \end{aligned}$$

$p_{XY}(-1,-1) = a\ , p_X(-1)p_Y(-1) = (2a+b)^2 \neq a$

Hence, X and Y are not independent.

$$m_X = \sum_i i p_X(i) = 0\ , m_Y = \sum_j j p_Y(j) = 0$$

$$\mu_{11} = \sum_i \sum_j ij p_{XY}(i,j) = (-1)(-1)a + (-1)(1)a + (1)(-1)a + (1)(1)a = 0$$

Hence,

$$\rho_{XY} = \frac{\mu_{11}}{\sigma_X \sigma_Y} = 0$$

4.27 (a) $m_Y = m_{X_1} + m_{X_2}$
$\sigma_Y^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2$

(b)
$$\mu_{X_2Y} = E\{(X_2 - m_{X_2})(Y - m_Y)\} = E\{(X_2 - m_{X_2})(X_1 - m_{X_1}) + (X_2 - m_{X_2})^2\} = \sigma_{X_2}^2$$

$$\rho_{X_2Y} = \frac{\mu_{X_2Y}}{\sigma_{X_2}\sigma_Y} = \frac{\sigma_{X_2}^2}{\sigma_{X_2}\sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2}} = \frac{\sigma_{X_2}}{\sqrt{\sigma_{X_1}^2 + \sigma_{X_2}^2}}$$

If $\sigma_{X_2}^2 \gg \sigma_{X_1}^2\ ,\ \rho_{X_2Y} \to 1$.

4.28 Let $Y_j = X_j^2$. Then

$$m_{Y_j} = E\{X_j^2\} = \int_{-\infty}^{\infty} x_j^2 f_{X_j}(x_j)dx_j = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx = 1$$

$$E\{Y_j^2\} = E\{X_j^4\} = \int_{-\infty}^{\infty} x_j^4 f_{X_j}(x_j)dx_j = 3$$

$$\sigma_{Y_j}^2 = E\{Y_j^2\} - m_{Y_j}^2 = 2$$

Now $Y = \sum_{j=1}^n Y_j$, where Y_j's are independent. Following Eqs. (4.38) and (4.41), we have

$$m_Y = \sum_{j=1}^n m_{Y_j} = n\ ,\ \sigma_Y^2 = \sum_{j=1}^n \sigma_{Y_j}^2 = 2n$$

4.29 (a) The first part is shown in the text [Eq. (4.41)]. Since $m_Y = m_{X_1} + m_{X_2} + \ldots + m_{X_n}$, we have

$$\begin{aligned} \mu = E\{(Y - m_Y)^3\} &= E\{[(X_1 - m_{X_1}) + (X_2 - m_{X_2}) + \ldots + (X_n - m_{X_n})]^3\} \\ &= E\left\{\sum_{j=1}^n (X_j - m_{X_j})^3 + \sum_{\substack{j,k \\ j\neq k}} (X_j - m_{X_j})(X_k - m_{X_k})^2\right\} \\ &= \sum_{j=1}^n \mu_j + \sum_{\substack{j,k \\ j\neq k}} E\{X_j - m_{X_j}\}E\{(X_k - m_{X_k})^2\} \\ &= \sum_{j=1}^n \mu_j \end{aligned}$$

Since $E\{X_j - m_{X_j}\} = 0$

(b) Consider $E\{(Y - m_Y)^4\}$.

$$E\{(Y - m_Y)^4\} = E\left\{[(X_1 - m_{X_1}) + \ldots + (X_n - m_{X_n})]^4\right\}$$
$$= E\left\{\sum_{j=1}^{n}(X_j - m_{X_j})^4\right\} + E\left\{\sum_{\substack{j,k\\ j\neq k}}(X_j - m_{X_j})^3(X_k - m_{X_k})\right\}$$
$$+ E\left\{\sum_{\substack{j,k\\ j\neq k}}(X_j - m_{X_j})^2(X_k - m_{X_k})^2\right\}$$
$$= \sum_{j=1}^{n} E\left\{(X_j - m_{X_j})^4\right\} + 0 + \sum_{\substack{j,k\\ j\neq k}} \sigma_{X_j}^2\sigma_{X_k}^2$$

Hence, $E\left\{(Y - m_Y)^4\right\} \neq \sum_{j=1}^{n} E\left\{(X_j - m_{X_j})^4\right\}$

Similar procedure can be used for higher-order moments.

4.30 (a) $\phi_X(t) = e^{jt(5)}p_X(5) = e^{5jt}$

$m_X = j^{-1}\phi'_X(0) = j^{-1}(5j) = 5$

$\alpha_2 = j^{-2}\phi''_X(0) = 25$

$\sigma_X^2 = \alpha_2 - m_X^2 = 0$

(b) $\phi_X(t) = e^{jt(5)}p_X(5) + e^{jt(7)}p_X(7) = \frac{1}{3}e^{5jt} + \frac{2}{3}e^{7jt}$

$m_X = j^{-1}\phi'_X(0) = \frac{19}{3}$

$\alpha_2 = j^{-2}\phi''_X(0) = \frac{123}{3}$

$\sigma_X^2 = \frac{123}{3} - \left(\frac{19}{3}\right)^2 = \frac{8}{9}$

(c) $\phi_X(t) = \sum_{k=1}^{\infty} \frac{1}{2^k}e^{jtk}$

$m_X = j^{-1}\phi'_X(0) = 2$

$\alpha_2 = j^{-2}\phi''_X(0) = 6$

$\sigma_X^2 = 6 - 2^2 = 2$

(d) $\phi_X(t) = \int_0^\infty e^{jtx}(ae^{-ax})dx = \frac{a}{a-jt}$

$m_X = j^{-1}\phi'_X(0) = \frac{1}{a}$

$\alpha_2 = j^{-2}\phi''_X(0) = \frac{2}{a^2}$

$\sigma_X^2 = \frac{2}{a^2} - \frac{1}{a^2} = \frac{1}{a^2}$

(e) $\phi_X(t) = 2\int_0^1 xe^{jtx}dx = \frac{2e^{jt}}{jt}\left(1 - \frac{1}{jt}\right) + \frac{2}{(jt)^2}$

$m_X = j^{-1}\phi'_X(0) = \frac{2}{3}$

$\alpha_2 = j^{-2}\phi''_X(0) = \frac{1}{2}$

$\sigma_X^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$

4.31 $M_X(t) = E\{e^{tX}\}$

$M'_X(t) = E\{Xe^{tX}\}$ and $M'_X(0) = \alpha_1$

$M''_X(t) = E\{X^2 e^{tX}\}$ and $M''_X(0) = \alpha_2$

Hence, $M_X^{(n)}(0) = \alpha_n \ , \ n = 1, 2, \ldots$

4.32 Let $Y_j = a_j X_j \ , \ j = 1, 2, \ldots, n$

$$\phi_{Yj}(t) = E\{e^{jtY_j}\} = E\{e^{j(ta_j)X_j}\} = \phi_{X_j}(a_j t)$$

Since, from Eq. (4.71),

$$\phi_Y(t) = \phi_{Y_1}(t)\phi_{Y_2}(t)\ldots\phi_{Y_n}(t)$$

we have

$$\phi_Y(t) = \phi_{X_1}(a_1 t)\phi_{X_2}(a_2 t)\ldots\phi_{X_n}(a_n t)$$