CHAPTER V

5.1 (a) $p_X(x) = \frac{1}{3}$, $x = 3$

$= \frac{2}{3}$, $x = 6$

Hence,

$$p_Y(3x-1) = \frac{1}{3}\ ,\ x = 3 \qquad \text{or} \qquad p_Y(y) = \frac{1}{3}\ ,\ y = 3(3) - 1 = 8$$

$$= \frac{2}{3}\ ,\ x = 6 \qquad\qquad = \frac{2}{3}\ ,\ y = 3(6) - 1 = 17$$

The PDF of Y is

$F_Y(y) = 0$, $y < 8$

$= \frac{1}{3}$, $8 \le y \le 17$

$= 1$, $y > 17$

(b) $F_Y(y) = P(Y \le y) = F_X[g^{-1}(y)] = F_X\left(\frac{y+1}{3}\right)$

Hence,

$$F_Y(y) = 0\ , \qquad y < 8$$

$$= \left(\frac{y+1}{3}\right)/3 - 1 = \frac{y-8}{9}\ ,\ 8 \le y \le 17$$

$$= 1\ , \qquad y > 17$$

5.2 $f_X(x) = \frac{1}{9}$, $86 \le x \le 95$

$= 0$, elsewhere

$C = 5(X-32)/9$, $g(x) = 5(x-32)/9$ and $g^{-1}(c) = 9c/5 + 32$

Since the transformation is monotone, Eq. (5.12) gives

$$f_C(c) = f_X[g^{-1}(c)]\left|\frac{dg^{-1}(c)}{dc}\right|$$

$$f_C(c) = \left(\frac{1}{9}\right)\left(\frac{9}{5}\right) = \frac{1}{5}\ ,\ \frac{5(86-32)}{9} \le c \le \frac{5(95-32)}{9} \text{ or } 30 \le c \le 35$$

$$= 0\ , \qquad \text{elsewhere}$$

5.3 $f_X(x) = 0 \ , \quad x < -1$
$= 1 + x \ , \ -1 \le x < 0$
$= 1 - x \ , \ 0 \le x < 1$
$= 0 \ , \quad x \ge 1$

$g(x) = 3x + 2 \ , g^{-1}(y) = (y-2)/3 \ , \dfrac{dg^{-1}(y)}{dy} = 1/3$

Eq. (5.12) gives

$$f_Y(y) = f_X[g^{-1}(y)]\left|\frac{dg^{-1}(y)}{dy}\right| = 0 \ , \qquad (y-2)/3 < -1 \text{ or } y < -1$$
$$= \frac{1}{3}[1 + (y-2)/3] = \frac{1}{9}(y+1) \ , \ -1 \le y < 2$$
$$= \frac{1}{3}[1 - (y-2)/3] = \frac{1}{9}(5-y) \ , \ 2 \le y < 5$$
$$= 0 \ , \qquad \text{elsewhere}$$

5.4 For $x \ge 0$

$$F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2) = F_X(y^2)$$

Hence,

$$F_Y(y) = F_X(y^2) \ , \ y \ge 0$$
$$= 0 \ , \quad y < 0$$

5.5 $f_X(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2} \ , -\infty < x < \infty$

$Y = e^X \ , g(x) = e^x \ , g^{-1}(y) = \ln y$

Eq. (5.12) gives

$$f_Y(y) = f_X[g^{-1}(y)\left|\frac{dg^{-1}(y)}{dy}\right| = \frac{1}{\sqrt{2\pi}} e^{-\ln^2 y/2}\left(\frac{1}{y}\right)$$
$$= \frac{1}{y\sqrt{2\pi}} e^{-\ln^2 y/2} \ , \ e^{-\infty} < y < e^{\infty} \text{ or } y > 0$$
$$= 0 \ , \qquad \text{elsewhere}$$

5.6 (a) $F_X(x) = \displaystyle\int_0^x f_X(u)du$

$$= 0 \ , \quad x < 0$$
$$= 1 - e^{-\lambda x} \ , \ x \ge 0$$

$$F_Y(y) = P\left[X \le \ln\left(\frac{y}{c}\right)\right] = F_X\left[\ln\left(\frac{y}{c}\right)\right]$$
$$= 0 \ , \quad y < c$$
$$= 1 - \left(\frac{y}{c}\right)^{-\lambda} \ , \ y \ge c$$

(b) $f_Y(y) = \dfrac{dF_Y(y)}{dy} = 0 \ , \quad y < c$

$$= \frac{\lambda}{c}\left(\frac{y}{c}\right)^{-(\lambda+1)} \ , \ y \ge c$$

5.7 $f_V(v) = \dfrac{1}{v_2 - v_1}$, $v_1 \le v \le v_2$

$= 0$, elsewhere

(a) $g(v) = ae^{b(v-c)^2}$, $g^{-1}(v) = c + \sqrt{\frac{\ln(r/a)}{b}}$

Since the transformation is monotone in the range (v_1, v_2) of v, Eq. (5.12) gives

$$f_R(r) = f_V[g^{-1}(r)]\frac{dg^{-1}(r)}{dr} = \left(\frac{1}{v_2 - v_1}\right)\left(\frac{1}{2r}\right)\frac{1}{\sqrt{b\ln(r/a)}} \ , \quad ae^{b(v_1-c)^2} \le r \le ae^{b(v_2-c)^2}$$
$$= 0 \ , \qquad \text{elsewhere}$$

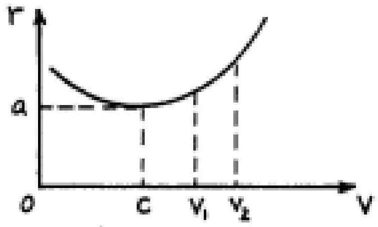

Figure 5.7a

(b) In this case, $g(v)$ admits two roots in the range (v_1, v_2) and

$$g_{1,2}^{-1}(r) = c \pm \sqrt{\frac{\ln(r/a)}{b}}$$
$$f_R(r) = f_V[g_1^{-1}(r)]\left|\frac{dg_1^{-1}(r)}{dr}\right| + f_V[g_2^{-1}(r)]\left|\frac{dg_2^{-1}(r)}{dr}\right|$$
$$= \left(\frac{1}{v_2 - v_1}\right)\frac{1}{r\sqrt{b\ln(r/a)}} \ , \quad a \le r \le ae^{b(v_2-c)^2}$$
$$= 0 \ , \qquad \text{elsewhere}$$

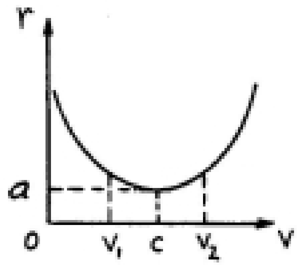

Figure 5.7b

5.8 $f_X(x) = \dfrac{1}{b - a}$, $a \le x \le b$

$= 0$, elsewhere

The given conditions imply that the transformation is monotone and non-decreasing. Hence, Eq. (5.12) gives

$$f_Y(y) = f_X[g^{-1}(y)]\left|\frac{dg^{-1}(y)}{dy}\right|$$

But $g^{-1}(y) = x$, the above is equivalent to

$$\frac{dy}{dx}f_Y(y) = f_X[g^{-1}(y)]$$

Now,

$$\frac{dy}{dx} = \frac{d[g(x)]}{dx} = g'[g^{-1}(y)]$$

Hence,

$$f_Y(y) = f_X[g^{-1}(y)]\left[1\Big/\left(\frac{dy}{dx}\right)\right] = \left(\frac{1}{b-a}\right)\frac{1}{g'[g^{-1}(y)]} \ , \quad g(a) \le y \le g(b)$$
$$= 0 \ , \qquad \text{elsewhere}$$

5.9 Let the projected area be denoted by X. The transformation is

$$\begin{aligned} X &= a\sin\Theta \\ f_\Theta(\theta) &= 2/\pi\ ,\ \ 0\le\theta\le\pi/2 \\ &= 0\ ,\quad \text{elsewhere} \end{aligned}$$

$g(\theta) = a\sin\theta\ ,\ g^{-1}(x) = \sin^{-1}(x/a)$

Since the transformation is monotone in the range $(0,\frac{\pi}{2})$ of Θ, Eq. (5.12) gives

$$\begin{aligned} f_X(x) &= f_\Theta[g^{-1}(x)]\left|\frac{dg^{-1}(x)}{dx}\right| \\ &= \left(\frac{2}{\pi}\right)\frac{1}{a\sqrt{1-(x/a)^2}} = \frac{2}{\pi\sqrt{a^2-x^2}}\ ,\ 0\le x\le a \\ &= 0\ , \qquad \text{elsewhere} \end{aligned}$$

5.10

$$\begin{aligned} f_V(v) = \frac{dF_V(v)}{dv} &= 0.19\left(\frac{v}{36.6}\right)^{-7.96}\exp\left[-\left(\frac{v}{36.6}\right)^{-6.96}\right]\ ,\ v>0 \\ &= 0\ , \qquad \text{elsewhere} \end{aligned}$$

$$g(v) = av^2\ ,\ g^{-1}(w) = \sqrt{w/a}\ ,\ \frac{dg^{-1}(w)}{dw} = \left(\frac{1}{2a}\right)\frac{1}{\sqrt{w/a}}$$

(a) Since V only takes positive values, the transformation $W = aV^2$ is monotone, Eq. (5.12) gives

$$\begin{aligned} f_W(w) &= f_V[g^{-1}(w)]\left|\frac{dg^{-1}(w)}{dw}\right| \\ &= \left(\frac{0.19}{2a}\right)\left(\frac{1}{\sqrt{w/a}}\right)\left(\frac{\sqrt{w/a}}{36.6}\right)^{-7.96}\exp\left[-\left(\frac{\sqrt{w/a}}{36.6}\right)^{-6.96}\right]\ ,\ w>0 \\ &= 0\ , \qquad \text{elsewhere} \end{aligned}$$

$$m_W = E\{W\} = \int_0^\infty w f_W(w)dw = 1.71\times 10^3 a$$

$$E\{W^2\} = \int_0^\infty w^2 f_W(w)dw = 3.74\times 10^6 a^2$$

$$\sigma_W^2 = E\{W^2\} - m_W^2 = 8.05\times 10^5 a^2$$

(b) $m_W = E\{W\} = E\{aV^2\} = a\int_0^\infty v^2 f_V(v)dv = 1.71\times 10^3 a$

$$E\{W^2\} = E\{a^2V^4\} = a^2\int_0^\infty v^4 f_V(v)dv = 3.74\times 10^6 a^2$$

$$\sigma_W^2 = E\{W^2\} - m_W^2 = 8.05\times 10^5 a^2$$

5.11 (a) $g(x) = |x|$

$g_1^{-1}(y) = -y\ ,\ g_2^{-1}(y) = y\ ,\ y>0$

Hence,

$$
\begin{aligned}
f_Y(y) &= f_X[g_1^{-1}(y)]\left|\frac{dg_1^{-1}(y)}{dy}\right| + f_X[g_2^{-1}(y)]\left|\frac{dg_2^{-1}(y)}{dy}\right| \\
&= \frac{1}{\sqrt{2\pi}}[e^{-(y+1)^2/2} + e^{-(y-1)^2/2}] \ , \ y > 0 \\
&= 0 \ , \qquad \text{elsewhere}
\end{aligned}
$$

$$m_Y = \int_0^\infty y f_Y(y)dy = \sqrt{\frac{2}{\pi}}e^{-1/2} + 2\ \text{erf}(1) = 1.1665$$

$$E\{Y^2\} = \int_0^\infty y^2 f_Y(y)dy = 2$$

$$\sigma_Y^2 = E\{Y^2\} - m_Y^2 = 2 - m_Y^2 = 0.6392$$

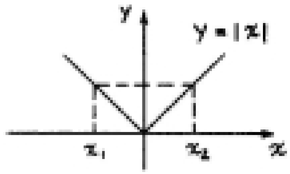

Figure 5.11

(b) $m_Y = E\{Y\} = E\{|X|\} = \int_{-\infty}^{\infty} |x| f_X(x)dx$

$$= \int_{-\infty}^{0} -x\left[\frac{1}{\sqrt{2\pi}}e^{-(x-1)^2/2}\right]dx + \int_0^\infty x\left[\frac{1}{\sqrt{2\pi}}e^{-(x-1)^2/2}\right]dx$$

$$= \sqrt{\frac{2}{\pi}}e^{-1/2} + 2\ \text{erf}(1) = 1.1665$$

$$E\{Y^2\} = E\{|X|^2\} = \int_{-\infty}^{\infty} x^2 f_X(x)dx = 2$$

$$\sigma_Y^2 = E\{Y^2\} - m_Y^2 = 2 - m_Y^2 = 0.6392$$

5.12 Since Y only assumes two discrete values (1 and 0), it is discrete with

$$
\begin{aligned}
p_Y(y) &= \int_0^\infty f_X(x)dx \ , y = 1 \\
&= \int_{-\infty}^0 f_X(x)dx \ , \ y = 0
\end{aligned}
$$

5.13 For $v > 0$, the transformation is monotone and Eq. (5.12) applies.

$$g(v) = mv^2/2 \ , \ g^{-1}(x) = \sqrt{2x/m}$$

$$
\begin{aligned}
f_X(x) &= f_V[g^{-1}(x)]\left|\frac{dg^{-1}(x)}{dx}\right| \\
&= a(2x/m)e^{-2bx/m}\left(\frac{1}{2\sqrt{2x/m}}\right) \\
&= \frac{a}{2}\sqrt{2x/m}e^{-2bx/m} \ , \quad \sqrt{2x/m} > 0 \text{ or } x > 0 \\
&= 0 \ , \qquad \text{elsewhere}
\end{aligned}
$$

5.14 (a) $A = 4\pi R^2$, $f_R(r) = \dfrac{1}{0.02r_0}$, $0.99r_0 \le r \le 1.01r_0$

$$= 0 \ , \qquad \text{elsewhere}$$

$$g(r) = 4\pi r^2 \ , g^{-1}(a) = \sqrt{a/4\pi}$$

The transformation is monotone in the range of R. Eq. (5.12) thus gives

$$
\begin{aligned}
f_A(a) &= f_R[g^{-1}(a)]\left|\frac{dg^{-1}(a)}{da}\right| \\
&= \left(\frac{1}{0.02r_0}\right)\frac{1}{4\sqrt{\pi a}} \\
&= \frac{1}{0.08r_0\sqrt{a\pi}}\ ,\ 4\pi(0.99r_0)^2 \le a \le 4\pi(1.01r_0)^2 \\
&= 0\ ,\qquad \text{elsewhere}
\end{aligned}
$$

(b) $V = \frac{4}{3}\pi R^3$, $g(r) = \frac{4}{3}\pi r^3$, $g^{-1}(v) = \left(\frac{3v}{4\pi}\right)^{1/3}$

$$
\begin{aligned}
f_V(v) &= f_R[g^{-1}(v)]\left|\frac{dg^{-1}(v)}{dv}\right| \\
&= \left(\frac{1}{0.02r_0}\right)\frac{1}{4\pi}\left(\frac{3v}{4\pi}\right)^{-2/3} \\
&= \frac{1}{0.08\pi r_o}\left(\frac{3v}{4\pi}\right)^{-2/3}\ ,\qquad \frac{4}{3}\pi(0.99r_0)^3 \le v \le \frac{4}{3}\pi(1.01r_0)^3 \\
&= 0\ ,\qquad \text{elsewhere}
\end{aligned}
$$

5.15 For both parts, the transformations are monotone in the range $r > 0$ of R. Eq. (5.12) thus holds.

(a) $g(r) = v/r$, $g^{-1}(i) = v/i$

$$
\begin{aligned}
f_I(i) &= f_R[g^{-1}(i)]\left|\frac{dg^{-1}(i)}{di}\right| \\
&= \frac{a^2v}{i}e^{-av/i}\left(\frac{v}{i^2}\right) \\
&= \frac{a^2v^2}{i^3}e^{-av/i}\ ,\ v/i > 0 \text{ or } i > 0 \\
&= 0\ ,\qquad \text{elsewhere}
\end{aligned}
$$

(b) $W = I^2R = Iv$, $g(i) = iv$, $g^{-1}(w) = w/v$

$$
\begin{aligned}
f_W(w) &= f_I[g^{-1}(w)]\left|\frac{dg^{-1}(w)}{dw}\right| \\
&= \frac{a^2v^2}{(w/v)^3}e^{-av/(w/v)}\left(\frac{1}{v}\right) \\
&= \frac{a^2v^4}{w^3}e^{-av^2/w}\ ,\qquad w/v > 0 \text{ or } w > 0 \\
&= 0\ ,\qquad \text{elsewhere}
\end{aligned}
$$

5.16 (a) (See Example 5.17)

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{X_1}(y - x_2)f_{X_2}(x_2)dx_2 \\
&= \int_{-1}^{1+y}(1/2)(1/2)dx_2 = \frac{2+y}{4}\ ,\ -2 \le y \le 0 \\
&= \int_{y-1}^{1}(1/2)(1/2)dx_2 = \frac{2-y}{4}\ ,\ \ 0 \le y \le 2
\end{aligned}
$$

(b) $\phi_{X_1}(t) = \phi_{X_2}(t) = \frac{1}{2}\int_{-1}^{1} e^{jtx}dx = \frac{\sin t}{t}$

$\phi_Y(t) = \phi_{X_1}(t)\phi_{X_2}(t) = \frac{\sin^2 t}{t^2}$

Thus,

$$f_Y(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jty}\phi_Y(t)dt = \frac{2+y}{4} \ , \ -2 \le y \le 0$$
$$= \frac{2-y}{4} \ , \ 0 \le y \le 2$$

5.17 (a) For $t > 0$,

$$P(T > t) = 4\int_0^{\infty}\int_{t+t_2}^{\infty} e^{-2(t_1+t_2)}dt_1dt_2$$
$$= \frac{1}{2}e^{-2t}$$

$$F_T(t) = 1 - P(T > t) = 1 - \frac{1}{2}e^{-2t} \ , \ t > 0$$

$$f_T(t) = \frac{dF_T(t)}{dt} = e^{-2t} \ , \ t > 0$$

Similarly, for $t \le 0$,

$$f_T(t) = e^{2t} \ , \ t \le 0$$

Hence,

$$f_T(t) = e^{-2|t|} \ , \ -\infty < t < \infty$$

(b) $m_T = m_{T_1} - m_{T_2} = \frac{1}{2} - \frac{1}{2} = 0$

$\sigma_T^2 = \sigma_{T_1}^2 + \sigma_{T_2}^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

5.18 The characteristic functions of X_1 and X_2 have been obtained in Example 4.14. They are

$$\phi_{X_1}(t) = [pe^{jt} + (1-p)]^{n_1}$$
$$\phi_{X_2}(t) = [pe^{jt} + (1-p)]^{n_2}$$

Then,

$$\phi_Y(t) = \phi_{X_1}(t)\phi_{X_2}(t) = [pe^{jt} + (1-p)]^{n_1+n_2}$$

By Comparing $\phi_Y(t)$ with ϕ_{X_1} or ϕ_{X_2}, it is clear that $\phi_Y(t)$ corresponds to a binomial distribution with parameters $(n_1 + n_2, p)$.

5.19 (a) Consider

$$F_Y(y) = P(Y \le y) = P(X_1 + X_2 \le y)$$
$$= P(X_1 = a \cap X_2 \le y - a) + P(X_1 = b \cap X_2 \le y - b)$$
$$= P(X_1 = a)F_{X_2}(y-a) + P(X_1 = b)F_{X_2}(y-b)$$
$$= pF_{X_2}(y-a) + qF_{X_2}(y-b)$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = pf_{X_2}(y-a) + qf_{X_2}(y-b)$$

or

$$f_Y(y) = pf_{Y_1}(y) + qf_{Y_2}(y)$$

where $Y_1 = X_2 + a$ and $Y_2 = X_2 + b$.

(b) $f_Y(y) = \frac{1}{3}\left[\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\right] + \frac{2}{3}\left[\frac{1}{\sqrt{2\pi}}e^{-(y-1)^2/2}\right] \ , \ -\infty < y < \infty$

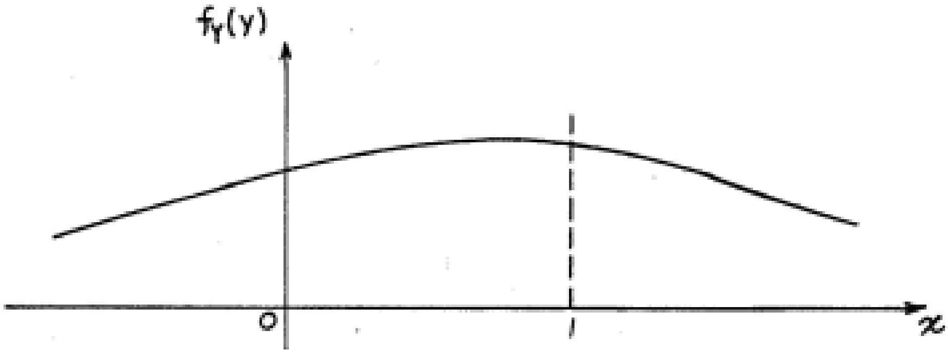

Figure 5.19

5.20 (a) For cold redundancy, $T_c = T_1 + T_2$. Eq. (5.53) gives

$$\begin{aligned}
F_{T_c}(t) &= \int_{-\infty}^{t}\int_{-\infty}^{t-t_2} f_{T_1}(t_1)f_{T_2}(t_2)dt_1dt_2 \\
&= \int_{-\infty}^{t} F_{T_1}(t-t_2)f_{T_2}(t_2)dt_2 \\
&= \int_{0}^{t} [1-e^{-a_1(t-t_2)}][a_2e^{-a_2t_2}]dt_2 \\
&= 1 - \frac{1}{a_1-a_2}(a_1e^{-a_2t} - a_2e^{-a_1t}) \ , \ t > 0 \\
&= 0 \ , \qquad \text{elsewhere}
\end{aligned}$$

$$\begin{aligned}
f_{T_c}(t) &= \frac{a_1a_2}{a_1-a_2}(e^{-a_2t} - e^{-a_1t}) \ , \qquad t > 0 \\
&= 0 \ , \qquad \text{elsewhere}
\end{aligned}$$

For hot redundancy, $T_H = \max(T_1, T_2)$

$$\begin{aligned}
F_{T_H}(t) &= P(T_H \le t) = P[\max(T_1,T_2) \le t] \\
&= P(T_1 \le t \cap T_2 \le t) = P(T_1 \le t)P(T_2 \le t) = F_{T_1}(t)F_{T_2}(t) \\
&= (1-e^{-a_1t})(1-e^{-a_2t}) \ , \ t > 0 \\
&= 0 \ , \qquad \text{elsewhere}
\end{aligned}$$

$$\begin{aligned}
f_{T_H}(t) &= a_1e^{-a_1t} + a_2e^{-a_2t} - (a_1+a_2)e^{-(a_1+a_2)t} \ , \ t > 0 \\
&= 0 \ , \qquad \text{elsewhere}
\end{aligned}$$

(b) Since $T_1 + T_2 \ge \max(T_1, T_2)$ for any non-negative values assumed by T_1 and T_2, we have

$$P[T_1 + T_2 \ge t] \ge P[\max(T_1,T_2) \ge t]$$

for every t. Hence, cold redundancy is preferred.

5.21 (See Example 5.15)

$T = \min(T_1, T_2)$

$$F_T(t) = P(T \le t) = P[\min(T_1, T_2) \le t] = P(T_1 \le t \cup T_2 \le t)$$

Using $P(A \cup B) = 1 - P(\bar{A}\bar{B})$, we have

$$F_T(t) = 1 - P(T_1 > t \cap T_2 > t) = 1 - P(T_1 > t)P(T_2 > t)$$
$$= 1 - [1 - F_{T_1}(t)][1 - F_{T_2}(t)]$$

$$f_T(t) = \frac{dF_T(t)}{dt} = f_{T_1}(t)[1 - F_{T_2}(t)] + f_{T_2}(t)[1 - F_{T_1}(t)]$$
$$= a_1 e^{-a_1 t}[1 - (1 - e^{-a_2 t})] + a_2 e^{-a_2 t}[1 - (1 - e^{-a_1 t})]$$
$$= (a_1 + a_2)e^{-(a_1+a_2)t} \ , \ t > 0$$
$$= 0 \ , \quad \text{elsewhere}$$

Generalizing, let $T = \min(T_1, T_2, \cdots, T_n)$

$$F_T(t) = 1 - P(T_1 > t \cap T_2 > t \cap \cdots \cap T_n > t)$$
$$= 1 - [1 - F_{T_1}(t)][1 - F_{T_2}(t)] \cdots [1 - F_{T_n}(t)]$$

$$f_T(t) = \frac{dF_T(t)}{dt} = f_{T_1}(t)[1 - F_{T_2}(t)] \cdots [1 - F_{T_n}(t)]$$
$$+ [1 - F_{T_1}(t)] f_{T_2}(t)[1 - F_{T_3}(t)] \cdots [1 - F_{T_n}(t)]$$
$$+ \cdots$$
$$+ [1 - F_{T_1}(t)][1 - F_{T_2}(t)] \cdots [1 - F_{T_{n-1}}(t)] f_{T_n}(t)$$
$$= (a_1 + a_2 + \cdots + a_n)e^{-(a_1+a_2+\cdots+a_n)t} \ , \quad t > 0$$
$$= 0 \ , \quad \text{elsewhere}$$

5.22 For $j \ge 0$,

$$p_Y(j) = P(Y = j) = P(X_2 - X_1 = j)$$
$$= \sum_{k=0}^{\infty} P(X_2 = j + k \cap X_1 = k) = \sum_{k=0}^{\infty} P(X_2 = j + k)P(X_1 = k)$$
$$= \sum_{k=0}^{\infty} \frac{\lambda^{j+k} e^{-\lambda}}{(j+k)!} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-2\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{2k+j}}{k!(k+j)!}$$
$$= e^{-2\lambda} I_j(2\lambda) \ , \ j = 0, 1, \ldots$$

where $I_j(x)$ is the modified Bessel function of order j.

Similarly, for $j < 0$,

$$p_Y(j) = e^{-2\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{2k-j}}{k!(k-j)!} \ , \ j = -1, -2, \ldots$$

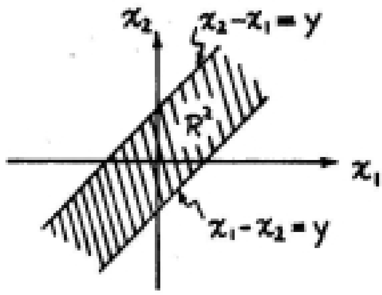

Figure 5.23

5.23 $$F_Y(y) = \iint_{[R^2: |x_1 - x_2| \le y]} f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2$$
$$= \int_{-\infty}^{\infty} \int_{x_2 - y}^{x_2 + y} f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \int_{-\infty}^{\infty} f_{X_2}(x_2)[f_{X_1}(x_2 + y) + f_{X_1}(x_2 - y)] dx_2 \ , \ -\infty < y < \infty$$

5.24 $$F_I(i) = \iint_{[R^2: c/x^2 \le i]} f_X(x) f_C(c) dc dx$$

The region R^2 for the intervals ($i \leq 16,\ 16 < i \leq 25\ ,25 < i \leq 64,\ 64 < i \leq 100$, and $i > 100$) are areas above the respective curves as shown in the figure. Hence,

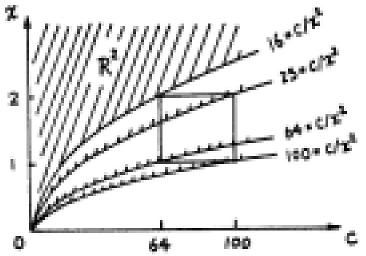

Figure 5.24

$$
\begin{aligned}
F_I(i) &= 0\ , & i \leq 16\\
&= \int_{8/\sqrt{i}}^{2}\int_{64}^{ix^2} \frac{1}{36} dcdx = \frac{1}{9}\left[\frac{256}{3\sqrt{i}} + \frac{2}{3}i - 32\right]\ , & 16 < i \leq 25\\
&= \int_{64}^{100}\int_{\sqrt{c/i}}^{2} \frac{1}{36} dxdc = 2 - \frac{244}{27\sqrt{i}}\ , & 25 < i \leq 64\\
&= 1 - \int_{1}^{\sqrt{100/i}}\int_{ix^2}^{100} \frac{1}{36} dcdx = \frac{1}{36}\left[136 - \frac{2000}{3\sqrt{i}} - \frac{i}{3}\right]\ , & 64 < i \leq 100\\
&= 1\ , & i > 100
\end{aligned}
$$

and

$$
\begin{aligned}
f_I(i) &= \frac{dF_I(i)}{di} = 0\ , & i \leq 16\\
&= \frac{2}{27}\left[1 - \frac{64}{i^{3/2}}\right]\ , & 16 < i \leq 25\\
&= \frac{122}{27 i^{3/2}}\ , & 25 < i \leq 64\\
&= \frac{1}{36}\left[\frac{1000}{3i^{3/2}} - \frac{1}{3}\right]\ , & 64 < i \leq 100\\
&= 0\ , & \text{otherwise}
\end{aligned}
$$

5.25 $F_Y(y) = P(Y \leq y) = P\left(\sqrt{X_1^2 + X_2^2} \leq y\right)$

$$= \int\int_{[R^2:\sqrt{x_1^2+x_2^2}\leq y]} f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2$$

$$\iint_{R^2} \frac{1}{2\pi} e^{-(x_1^2+x_2^2)/2} dx_1 dx_2$$

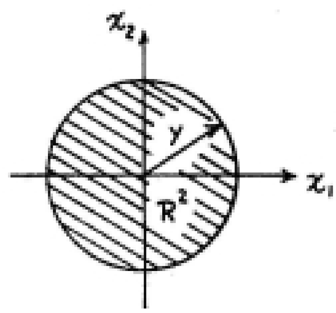

Figure 5.25

Using polar coordinates: $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \tan^{-1}(x_2/x_1)$, we have

$$F_Y(y) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{y} e^{-r^2/2} r dr d\theta = 1 - e^{-y^2/2}\ ,\ y \geq 0$$

$$
\begin{aligned}
f_Y(y) &= \frac{dF_Y(y)}{dy} = y e^{-y^2/2}\ ,\ y \geq 0\\
&= 0\ , \qquad \text{elsewhere}
\end{aligned}
$$

5.26 $F_Y(y) = \underset{[R^2:\sqrt{x_1^2+x_2^2+x_3^2}\leq y]}{\iiint} f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3) dx_1 dx_2 dx_3$

Using spherical coordinates (r, θ, ϕ),

$dx_1 dx_2 dx_3 = r^2 \sin\theta d\phi d\theta dr$

and

$$F_Y(y) = \frac{1}{(2\pi)^{3/2}} \underset{R^3}{\iiint} e^{-(x_1^2+x_2^2+x_3^2)/2} dx_1 dx_2 dx_3$$

$$= \frac{1}{(2\pi)^{3/2}} \int_0^y \int_0^{\pi} \int_0^{2\pi} e^{-r^2/2} r^2 \sin\theta d\phi d\theta dr$$

$$= \frac{4\pi}{(2\pi)^{3/2}} \int_0^y r^2 e^{-r^2/2} dr \ , \ y > 0$$

$$f_Y(y) = \sqrt{\frac{2}{\pi}} y^2 e^{-y^2/2} \ , \ y > 0$$

$= 0$, elsewhere

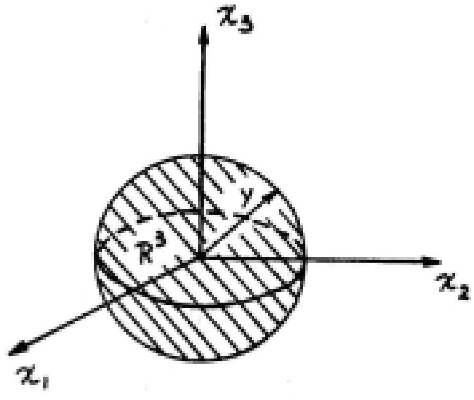

Figure 5.26

5.27 Let

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \text{and} \quad \underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} X_1 + X_2 + X_3 \\ X_1 \\ X_2 \end{bmatrix}$$

Then $g_1(\underline{x}) = x_1 + x_2 + x_3$, $g_2(\underline{x}) = x_1$, $g_3(\underline{x}) = x_2$

$g_1^{-1}(\underline{y}) = y_2$, $g_2^{-1}(\underline{y}) = y_3$, $g_3^{-1}(\underline{y}) = y_1 - y_2 - y_3$

According to Eq. (5.67), we have

$f_{\underline{Y}}(\underline{y}) = f_{\underline{X}}[\underline{g}^{-1}(\underline{y})]|J|$

where

$$J = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} = 1$$

$$f_Y(y) = \frac{6}{(1 + y_2 + y_3 + y_1 - y_2 - y_3)^4} = \frac{6}{(1+y_1)^4} \ , \ y_1 \geq 0 \ , \ 0 \leq y_2 \leq y \ , \ 0 \leq y_3 \leq y_1 - y_2$$

$= 0$, elsewhere

The desired pdf is $f_{Y_1}(y_1)$ and we have

$$f_{Y_1}(y_1) = \int_0^{y_1-y_2} \int_0^{y_1} f_{\underline{Y}}(\underline{y}) dy_2 dy_3 = \frac{3y_1^2}{(1+y_1)^4} \ , \ y_1 > 0$$

$= 0$, elsewhere

5.28 $g_1(x_1, x_2) = x_1 + x_2$, $g_2(x_1, x_2) = x_1/(x_1 + x_2)$

$$g_1^{-1}(y_1, y_2) = y_1 y_2 \ , \quad \frac{\partial g_1^{-1}}{\partial y_1} = y_2 \ , \quad \frac{\partial g_1^{-1}}{\partial y_2} = y_1$$

$$g_2^{-1}(y_1, y_2) = y_1 - y_1 y_2 \ , \ \frac{\partial g_2^{-1}}{\partial y_1} = 1 - y_2 \ , \ \frac{\partial g_2^{-1}}{\partial y_2} = -y_1$$

$$J = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = -y_1$$

Hence, Eq. (5.67) gives

$$f_{Y_1Y_2}(y_1,y_2) = f_{X_1X_2}(g_1^{-1}, g_2^{-1})|J|$$
$$= e^{-y_1y_2}e^{-(y_1-y_1y_2)}(y_1)\ ,\ y_1y_2 \geq 0,\ y_1 - y_1y_2 \geq 0 \text{ or } y_1 > 0,\ 0 \leq y_2 \leq 1$$
$$= 0\ , \quad \text{elsewhere}$$

Hence,

$$f_{Y_1Y_2}(y_1,y_2) = y_1e^{-y_1}\ ,\ y_1 > 0\ ,\ 0 \leq y_2 \leq 1$$
$$= 0\ , \quad \text{elsewhere}$$

We see that $f_{Y_1Y_2}(y_1,y_2)$ is in the form of $f_{Y_1}(y_1)f_{Y_2}(y_2)$ with

$$f_{Y_1}(y_1) = y_1e^{-y_1}\ ,\ y_1 > 0 \qquad f_{Y_2}(y_2) = 1\ ,\ 0 \leq y_2 \leq 1$$
$$= 0\ , \quad \text{elsewhere} \qquad \text{and} \qquad = 0\ , \quad \text{elsewhere}$$

Y_1 and Y_2 are thus independent.

5.29 Let

$$\underline{W} = \begin{bmatrix} X \\ Y \end{bmatrix} \quad \text{and} \quad \underline{Z} = \begin{bmatrix} R \\ \Phi \end{bmatrix}$$

$$g_1(\underline{w}) = \sqrt{x^2+y^2}\ ,\ g_2(\underline{w}) = \tan^{-1}(y/x)$$
$$g_1^{-1}(\underline{z}) = r\cos\phi\ ,\ g_2^{-1}(\underline{z}) = r\sin\phi$$

It follows from Eq. (5.67) that

$$f_{\underline{Z}}(z) = f_{\underline{W}}[\underline{g}^{-1}(\underline{z})]|J|$$

where

$$J = \begin{vmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{vmatrix} = r$$

Hence

$$f_{\underline{Z}}(z) = \frac{1}{2\pi\sigma^2}e^{-(r^2\cos^2\phi + r^2\sin^2\phi)/2\sigma^2}(r)$$
$$= \frac{r}{2\pi\sigma^2}e^{-r^2/2\sigma^2}\ ,\ r \geq 0\ ,\ -\pi \leq \phi \leq \pi$$
$$= 0\ , \quad \text{elsewhere}$$

By inspection, the marginal pdf's of R and Φ are

$$f_R(r) = \frac{r}{\sigma^2}e^{-r^2/2\sigma^2}\ ,\ r \geq 0$$
$$= 0\ , \quad \text{elsewhere}$$

$$f_\Phi(\phi) = \frac{1}{2\pi}\ , \quad -\pi \leq \phi \leq \pi$$
$$= 0\ , \quad \text{elsewhere}$$

Since $f_Z(z) = f_R(r)f_\Phi(\phi)$, the r.v.'s R and Φ are independent.

5.30 In place of Eq. (5.64) one can write

$$\int \underset{R^n_{\underline{Y}}}{\cdots} \int f_Y(\underline{y})d\underline{y} = \int \underset{R^n_{\underline{X}}}{\cdots} \int f_{\underline{Y}}[\underline{g}(\underline{x})]|J'|dx$$

Using Eq. (5.63), we have

$$f_{\underline{Y}}[\underline{g}(\underline{x})] = f_{\underline{X}}(\underline{x})|J'|^{-1}$$

or

$$f_{\underline{Y}}(\underline{y}) = f_{\underline{X}}[\underline{g}^{-1}(\underline{y})]|J'|^{-1}$$

where J' is evaluated at $\underline{x} = \underline{g}^{-1}(\underline{y})$