CHAPTER VI

6.1 Following a similar procedure as that in Example 4.17, we have

$$\begin{aligned}\phi_X(t) &= (pe^{jt}+q)^n \\ &= \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} e^{jit}\end{aligned}$$

Comparing the above with the definition of characteristic function given in Eq. (4.46), we have

$$p_X(k) = \binom{n}{k} p^k q^{n-k} \ , \ \ k = 0, 1, 2, \ldots, n$$

The mean and variance of X are directly derivable from Eq. (6.10). According to Eqs. (4.38) and (4.41), we have

$$\begin{aligned}&m_X = m_{X_1} + \cdots + m_{X_n} = np \\ &\sigma_X^2 = \sigma_{X_1}^2 + \cdots + \sigma_{X_2}^2 = npq\end{aligned}$$

6.2 $m_X = 240$, $\sigma_X^2 = 48$

For binomial distribution, $m_X = np$ and $\sigma_X^2 = np(1-p)$. Hence,

$np = 240$ and $np(1-p) = 48$

Solving for n and p, we have $n = 300$ and $p = 0.8$. Then

$$p_X(0) = \binom{n}{0} p^0 (1-p)^{n-0} = (1-p)^n = (0.2)^{300} \cong 0$$

6.3 Let X be the number of successful experiments. It is then binomial with $n = 5$, $p = 0.75$ (probability of success).

(a) $p_X(5) = p^5 = (0.75)^5 = 0.237$

(b) $m_X = np = (5)(0.75) = 3.75$ or 3.75 experiments are expected to succeed.

6.4 Let X be the number of unsafe days in a 30-day month. It is then binomial with $n = 30$, $p = 0.2$ (unsafe).

(a) $p_X(7) = \binom{30}{7} (0.2)^7 (0.8)^{30-7} = 0.154$

(b) $m_X = np = (30)(0.2) = 6$ days

6.5 (a) Let X = number of ticket holders showing up. Then X is binomial with $n = 84$, $p = 0.95$ (showing up).

$$
\begin{aligned}
P(X \leq 80) &= 1 - \sum_{k=81}^{84} p_X(k) \\
&= 1 - \binom{84}{81} 0.95^{81} 0.05^3 - \binom{84}{82} 0.95^{82} 0.05^2 - \binom{84}{83} 0.95^{83} 0.05 - 0.95^{84} \\
&= 0.611
\end{aligned}
$$

(b) Let Y = number of no shows.

$E\{Y\} = nq = (84)(0.05) = 4.2 \cong 4$

6.6 (a) Let X be the number of boys. It is then binomial with $n = 4$, $p = 0.51$ (boy).

(i) $p_X(1) = \binom{4}{1}(0.51)(0.49)^3 = 0.24$

(ii) $p_X(3) = \binom{4}{3}(0.51)^3(0.49) = 0.26$

(iii) $\sum_{j=1}^{4} p_X(j) = 1 - p_X(0) = 1 - \binom{4}{0}(0.49)^4 = 0.942$

(iv) $\sum_{j=0}^{3} p_X(j) = 1 - p_X(4) = 1 - \binom{4}{4}(0.51)^4 = 0.932$

(b) Determine n such that $1 - p_X(0) - p_X(1) > 0.75$. Consider

$$
\begin{aligned}
x &= 1 - \binom{n}{0}(0.49)^n - \binom{n}{1}(0.51)(0.49)^{n-1} \\
&= 1 - (0.49)^n - n(0.51)(0.49)^{n-1}
\end{aligned}
$$

For $n = 4$, $x = 0.702$

$n = 5$, $x = 0.825$

Hence, $n = 5$

6.7 (a) X is $B(n,p)$ with $n = 5$ and $p = 0.25$

(b) $P(X \leq 3) = 1 - \sum_{k=4}^{5} \binom{n}{k} p^k q^{n-k} \cong 0.99$

6.8 (a) Let X = number of n permit holders wanting a parking space. Then

$$
\begin{aligned}
P(X > m) = 1 - P(X \leq m) &= 1 - \sum_{k=0}^{m} p_X(k) \\
&= 1 - \sum_{k=0}^{m} \binom{n}{k} p^k (1-p)^{n-k}
\end{aligned}
$$

(b) Let Y = number of people turned away. Then

$$P(Y = 0) = P(X \leq m) = \sum_{k=0}^{m} p_X(k)$$

$$P(Y=1) = P(X=m+1) = p_X(m+1)$$
$$\cdots$$
$$P(Y=n-m) = P(X=n) = P_X(n)$$
$$\begin{aligned} E\{Y\} &= (0)P(Y=0) + (1)P(Y=1) + \cdots + (n-m)P(Y=n-m) \\ &= p_X(m+1) + 2p_X(m+2) + \cdots + (n-m)p_X(n) \\ &= \sum_{k=m+1}^{n} (k-m)p_X(k) = \sum_{k=m+1}^{n} (k-m)\binom{n}{k} p^k(1-p)^{n-k} \end{aligned}$$

6.9 Binomial approximation is justified as $n \to \infty$ from the following observation: The binomial distribution is applicable to the case of sampling with replacement, while the hypergeometric distribution is applicable to the case of sampling without replacement. As $n \to \infty$, the results differ insignificantly whether or not a particular sampled item is returned to the lot before the next one is chosen.

6.10 Let X = number of defective parts in the 10-part sample. Then X has a hypergeometric distribution with $n = 100$, $n_1 = 5$ and $m = 10$. Using Eq. (6.13), we have

$$P(X=0) = p_X(0) = \frac{\binom{5}{0}\binom{95}{10}}{\binom{100}{10}} = 0.584$$

6.11 (a) Z has a hypergeometric distribution defined by Eq. (6.13) with $n = 10$, $n_1 = 2$ and $m = 4$. Hence,

$$p_Z(k) = \frac{\binom{2}{k}\binom{8}{4-k}}{\binom{10}{4}}\ , \ k = 0, 1, 2$$

or

$$p_Z(k) = \begin{cases} 5/15\ , & k = 0 \\ 8/15\ , & k = 1 \\ 2/15\ , & k = 2 \end{cases}$$

(b) $P(Z \geq 1) = 1 - p_Z(0) = 2/3$

(c) $$\begin{aligned} P(Z \geq 1) &= 2/3\ , \quad m = 4 \\ &= 0.778\ , \ m = 5 \end{aligned}$$

Hence, five must be selected.

6.12 Let X = number of trials necessary for the rth success to occur. Then X has a negative binomial distribution.

$$P(X < s+r) = \sum_{k=r}^{s+r-1} p_X(k) = \sum_{k=r}^{s+r-1} \binom{k-1}{r-1} p^r(1-p)^{k-r}$$

6.13 (See Example 6.8). Let X be the number of trials to the fifth success (turning left). Then X has a negative binomial distribution with $r = 5$ and $p = 0.25$.

$$\begin{aligned} P(X \leq 10) &= \sum_{k=5}^{10} p_X(k) = \sum_{k=5}^{10} \binom{k-1}{4} (0.25)^5(0.75)^{k-5} \\ &= (0.25)^5(1 + 3.75 + 8.438 + 14.766 + 22.148 + 29.9) \\ &= 0.078 \end{aligned}$$

6.14 Let X be the number of successes in n steps. Then X is binomial with $n = 10$ and $p = 0.99$. Hence,

$$P(X = 10) = \binom{10}{10}(0.99)^{10}$$

and

$$P(\text{contamination}) = 1 - P(X = 10) = 1 - 0.99^{10} = 0.096$$

6.15 (a) X is negative binomial with $r = 3$ and $p = 0.7$. Hence,

$$P(X \leq 6) = \sum_{k=3}^{6} \binom{k-1}{r-1} p^r q^{k-r}$$

(b) $E\{X - r\} = E\{X\} - r = \dfrac{r}{p} - r$

(c) $P(FFFSSS \cup SFFSSS \cup SSFSSS)$
$= P(FFFSSS) + P(SFFSSS) + P(SSFSSS)$
$= p^3(q^3 + pq^2 + p^2q)$

6.16 Let X = number of floods equal to or exceeds the "100-yr" flood in 100 years. Then X is binomial with $n = 100$ and $p = 1/100 = 0.01$.

(a) $p_X(1) = \binom{100}{1}(0.01)(0.99)^{99} = 0.37$

(b) $P(X \geq 1) = 1 - p_X(0) = 1 - \binom{100}{0}(0.99)^{100} = 0.634$

6.17 (a) Let X be number of trials until the first defective part is found. Then X has a geometric distribution with $p = 0.1$ (defective).

$$P(X > 10) = \sum_{k=11}^{\infty} q^{k-1}p = q^{10}p\left(\sum_{k=0}^{\infty} q^k\right) = \frac{q^{10}p}{1-q}$$
$$= q^{10} = 0.9^{10} = 0.349$$

(b) $p = 0.25$, $P(X > k) = q^k$ and $P(X \leq k) = 1 - q^k$
It is required that

$$1 - (0.75)^k \geq 0.75 \qquad \text{or} \qquad \frac{\ln 0.25}{\ln 0.75} = 4.813 \cong 5$$

6.18 (a) X_1, X_2 and X_3 have a multinomial distribution with $n = 8$, $p_1 = 0.4$, $p_2 = 0.1$ and $p_3 = 0.5$.

(b) $P(X_1 = 4 \cap X_2 = 4) = p_{X_1X_2X_3}(4, 4, 0) = \dfrac{8!}{4!4!}(0.4)^4(0.1)^4 = 0.00018$

(c) X_1 is $B(n, p)$ with $n = 8$ and $p = 0.4$.

$$P(X_1 > 4) = 1 - \sum_{k=5}^{8} \binom{n}{p} p^k q^{n-k}$$

6.19

$$\begin{aligned} p_{X_1X_2}(k_1,k_2) &= P_{X_1X_2X_3}(k_1,k_2,10-k_1-k_2) \\ &= \frac{10!}{k_1!k_2!(10-k_1-k_2)!}(0.1)^{k_1}(0.7)^{k_2}(0.2)^{10-k_1-k_2} \ , \\ &\qquad\qquad k_1,k_2 = 0,1,2,\dots \text{ and } k_1+k_2 \le 10 \end{aligned}$$

$$\begin{aligned} P(X_1 \le 1 \cap X_2 \le 2) = F_{X_1X_2}(1,2) &= \sum_{k_1=0}^{1}\sum_{k_2=0}^{2} p_{X_1X_2}(k_1,k_2) \\ &= p_{X_1X_2}(0,0)+p_{X_1X_2}(1,0)+p_{X_1X_2}(0,1)+p_{X_1X_2}(1,1)+p_{X_1X_2}(0,2)+p_{X_1X_2}(1,2) \\ &= 3.55\times 10^{-4} \end{aligned}$$

6.20 Let Y be the number of trials required for the first successful countdown. Then Y has a geometric distribution with pmf

$$p_Y(j) = q^{j-1}p \ , \quad j = 1,2,\cdots \tag{1}$$

Suppose now we label the pads 1 and 2 and the first vehicle on each pad by the same number, the standby vehicle being labeled 3. Then there is only one possible way in which the vehicles may be launched.

Specifically, the order of launching is 1-2-3. However, there are two distinct and mutually exclusive cases to consider, there are

Case	**Event**
1.	#2 goes before turnaround time is completed
2.	#2 goes on or after turnaround time is completed

Now the probability of three successful countdowns in k days given case 1 is

$$P\left\{k \text{ days} \,\middle|\, \text{Case } 1\right\} = \sum_{n=1}^{k-r-1}\sum_{m=1}^{r-1} P\left\{\begin{matrix}\#1 \text{ took } n \text{ trials, } \#2 \text{ took } m \text{ trials.} \\ \text{and } \#3 \text{ took } k-n-r \text{ trials.}\end{matrix}\right\} \tag{2}$$

where the limits for m and n are easily understood from the following figures:

$\longleftarrow$ k days $\longrightarrow$		
n days	r days	$k-n-r$ days
#1	#2	#3

$1 \le m \le r-1$ trials at most available to #2

$\longleftarrow$ k days $\longrightarrow$		
n days	r days	1 day
#1	#2	#3

$1 \le n \le k-r-1$ trials at most available to #1

From the assumed independence of the vehicles, the joint event appearing in the summation of Eq. (2) can be written in the form

$$P\left\{k \text{ days} \,\middle|\, \text{Case } 1\right\} = \sum_{n=1}^{k-r-1}\sum_{m=1}^{r-1} pq^{n-1}pq^{m-1}pq^{k-n-r-1} \tag{3}$$

$$= \sum_{n=1}^{k-r-1} p^2q^{k-r-2}\sum_{m=1}^{r-1} pq^{m-1}$$

$$= \sum_{n=1}^{k-r-1} p^2q^{k-r-2}\left[1-q^{r-1}\right] \tag{4}$$

$$= p^2q^{k-r-2}(1-q^{r-1})(k-r-1)$$

Similarly, for case 2 we have

$$P\left\{k \text{ days} \,\middle|\, \text{Case 2}\right\} = \sum_{n=1}^{k-r-1} \sum_{m=r}^{k-n-1} P\left\{\begin{array}{l}\#1 \text{ took } n \text{ trials, } \#2 \text{ took } m \text{ trials,} \\ \text{and } \#3 \text{ took } k-n-m \text{ trials.}\end{array}\right\} \tag{5}$$

where, as before, the limits for m and n are easily deduced from a consideration of the following figure:

$\longleftarrow$ k days $\longrightarrow$

n days	r days	$m-r$ days	$k-n-m$ days
#1		#2	#3

$r \le m \le k-n-1$ trials at most available to #2

$\longleftarrow$ k days $\longrightarrow$

n days	$r-1$ days	1 day	1 day
#1		#2	#3

$1 \le n \le k-r-1$ trials at most available to #1

Thus,

$$\begin{aligned}
P\left\{k \text{ days} \,\middle|\, \text{Case 2}\right\} &= \sum_{n=1}^{k-r-1} \sum_{m=r}^{k-n-1} p_n p_m p_{k-n-m} \\
&= \sum_{n=1}^{k-r-1} pq^{n-1} \sum_{m=r}^{k-n-1} pq^{m-1} pq^{k-n-m-1} && (6) \\
&= \sum_{n=1}^{k-r-1} p^3 q^{k-3} (k-n-r) \\
&= p^3 q^{k-3} \left[(k-r)(k-r-1) - \frac{(k-r)(k-r-1)}{2}\right] && (7) \\
&= \frac{p^3 q^{k-3}}{2} (k-r)(k-r-1)
\end{aligned}$$

The sum of Eqs. (4) and (7) gives $p_X(k)$ we are seeking.

6.21 $E\{X^2(0,t)\} = \sum_{k=0}^{\infty} k^2 \frac{(\lambda t)^k e^{-\lambda t}}{k!} = \sum_{k=1}^{\infty} \frac{k(\lambda t)^k e^{-\lambda t}}{(k-1)!}$

Let $j = k-1$, we have

$$\begin{aligned}
E\{X^2(0,t)\} &= \sum_{j=0}^{\infty} (j+1) \frac{(\lambda t)^{j+1} e^{-\lambda t}}{j!} \\
&= \lambda t \sum_{j=0}^{\infty} j \frac{(\lambda t)^j e^{-\lambda t}}{j!} + \lambda t \sum_{j=0}^{\infty} \frac{(\lambda t)^j e^{-\lambda t}}{j!} \\
&= \lambda t(\lambda t) + \lambda t(1) = (\lambda t)^2 + \lambda t
\end{aligned}$$

$$\sigma^2_{X(0,t)} = E\{X^2(0,t)\} - m_{X(o,t)} = (\lambda t)^2 + (\lambda t) - (\lambda t)^2 = \lambda t$$

6.22 Let us form the ratio

$$\frac{p_k(0,t)}{p_{k-1}(0,t)} = \frac{(\lambda t)^k e^{-\lambda t}/k!}{(\lambda t)^{k-1} e^{-\lambda t}/(k-1)!} = \frac{\lambda t}{k}\ , \ \ k = 1, 2, \ldots$$

Thus, $p_k(0,t)$ increases when $k < \lambda t$, reaching its maximum value when k is the largest integer not exceeding λt.

6.23 Let X be the number of accidents in a month. Then X is Poisson with $\lambda = 1/2$.

(a) $E\{\text{No. of accidents per year}\} = 12E\{X\} = 12\left(\frac{1}{2}\right) = 6$

(b) $P(X = 0) = \left(\frac{1}{2}\right)^0 e^{-1/2}/0! = e^{-1/2}$

6.24 $\nu = \lambda t = (3.25)(4) = 13$

From Fig. 6.4, we have

$P(X \le 9) \cong 0.166$

6.25 X is Poisson with $\lambda t = 0.5(10) = 5$.

(a) $P(X \ge 4) = 1 - P(X < 4) = 1 - \sum_{k=0}^{3} \frac{5^k e^{-5}}{k!} = 0.71$

(b) Let Y be number of intervals in 7 having 4 or more emitted particles. Then Y is binomial with $n = 7$ and $p = 0.71$. Hence,

$$P(Y \ge 1) = 1 - P(Y = 0) = 1 - \binom{7}{0}(0.71)^0(0.29)^7 = 0.9998$$

6.26 $\nu = \lambda t = (0.5)(60) = 30$

$P[20 < X(0,t) \le 40] = P(X \le 40) - P(X \le 20)$

From Fig. 6.4, we have

$P[20 < X(0,t) \le 40] \cong 0.97 - 0.035 = 0.935$

6.27 Let X be the actual count. Then

$$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}\ ,\ k = 0, 1, 2$$

Hence,

$$p_Y(k) = \frac{\lambda^k e^{-\lambda}}{k!}\ ,\ k = 0, 1, 2, \ldots, 29$$

$$p_Y(30) = \sum_{k=30}^{\infty} p_X(k) = 1 - \sum_{k=0}^{29} \frac{\lambda^k e^{-\lambda}}{k!}$$

6.28 Let X be the number of students having birthday on a given day. Then X is binomial with $n = 200$ and $p = 1/365$ (birthday on any given day).

$$p_X(20) = \binom{200}{20}\left(\frac{1}{365}\right)^{20}\left(\frac{364}{365}\right)^{200-20}$$

Using Poisson approximation, $\nu = np = \frac{200}{365} = 0.548$

$$p_X(20) \cong \frac{(0.548)^{20} e^{-0.548}}{20!} = 1.4 \times 10^{-24}$$

6.29 (a) Let X be the number of misprints on a given page. Then

$$p_X(k) = \frac{\nu^k e^{-\nu}}{k!} \ , \ \nu = 1 \text{ and } k = 0, 1, 2, \ldots$$

Hence, $P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-1} = 0.633$

(b) Let Y be the number of pages with at least one misprint. Then

$$p_Y(k) = \binom{n}{k} p^k (1-p)^{n-k} \ , \ k = 0, 1, 2, \ldots, 500 \, ,$$

with $n = 500$ and $p = 0.633$.

$$\begin{aligned} P(Y \geq 3) = 1 - \sum_{j=0}^{2} p_Y(j) &= 1 - \binom{500}{0}(0.633)^0(0.367)^{500} \\ &\quad - \binom{500}{1}(0.633)(0.367)^{499} - \binom{500}{2}(0.633)^2(0.367)^{498} \\ &\cong 1 \end{aligned}$$

6.30 (See Example 6.13) X has Poisson distribution with

$$p_X(k) = \frac{(p\nu)^k e^{-p\nu}}{k!} \ , \ k = 0, 1, 2, \ldots$$

where $p = 0.09$ and $\nu = 250$. $p\nu = (0.09)(250) = 22.5$. Hence,

$$p_X(k) = \frac{22.5^k e^{-22.5}}{k!} \ , \ k = 0, 1, 2, \ldots$$

6.31

$$\begin{aligned} p_X(k) &= \int_0^\infty p_{X\Lambda}(k|\lambda) f_\Lambda(\lambda) d\lambda \\ &= \frac{t^k (a/b)^a}{\Gamma(a) k!} \int_0^\infty \lambda^{a+k-1} e^{-(t+\frac{a}{b})\lambda} d\lambda \end{aligned}$$

Using $\Gamma(c) = \int_0^\infty x^{c-1} e^{-x} dx$,

$$\begin{aligned} p_X(k) &= \frac{t^k (a/b)^a}{\Gamma(a) k!} \left[\frac{\Gamma(a+k)}{(t + \frac{a}{b})^{a+k}} \right] \\ &= \frac{\Gamma(a+k)}{k! \Gamma(a)} \left[\frac{a}{a+bt} \right]^a \left[\frac{bt}{a+bt} \right]^k \ , \ k = 0, 1, 2, \ldots \end{aligned}$$

6.32 The differential equation for $p_k(0,t)$ in this case is

$$\frac{dp_0(0,t)}{dt} = -(v/w) t^{\nu-1} p_0(0,t) \ , \ p_0(0,0) = 1$$

and

$$\frac{dp_k(0,t)}{dt} = -(v/w) t^{\nu-1} [p_k(0,t) - p_{k-1}(0,t)] \ ,$$

$p_k(0,0) = 1 \ , \ k = 1, 2, \ldots$

The solution of the above yields

$$p_k(0,t) = \frac{(t^v/w)^k \exp(-t^v/w)}{k!} \ , \ k = 0, 1, 2, \ldots$$

6.33

$$\begin{aligned} p_{X_1X_2}(j,k) &= P[X_2 = k | X_1 = j] P(X_1 = j) \\ &= P[X(t_1, t_2) = k - j] P(X_1 = j) \\ &= \left\{ \frac{[\lambda(t_2 - t_1)]^{k-j} e^{-\lambda(t_2 - t_1)}}{(k-j)!} \right\} \left\{ \frac{(\lambda t_1)^j e^{-\lambda t_1}}{j!} \right\} \\ &= \frac{(\lambda t_1)^j [\lambda(t_2 - t_1)]^{k-j} e^{-\lambda t_2}}{j!(k-j)!} \ , \ j, k = 0, 1, 2, \ldots \text{and } k \geq j \end{aligned}$$

$$P(X_1 \leq \lambda t_1 \cap X_2 \leq \lambda t_2) = F_{X_1X_2}(\lambda t_1, \lambda t_2) = \sum_{j=0}^{n} \sum_{k=j}^{m} p_{X_1X_2}(j,k)$$

where n and m are, respectively, the largest integers less than or equal to λt_1 and λt_2.