CHAPTER VII

7.1 See the solution to Problem 3.23.

7.2 (a) $Y = aX$

$$\phi_Y(t) = E\{e^{jtY}\} = E\{e^{j(at)X}\} = \phi_X(at) = \frac{\sin at}{at}$$

(b) $Y = a + \frac{b}{2} + \frac{b}{2}X$

$$\begin{aligned}\phi_Y(t) = E\{e^{jtY}\} &= E\left\{e^{jt\left(a+\frac{b}{2}\right)}e^{j\left(\frac{bt}{2}\right)X}\right\} \\ &= e^{jt\left(a+\frac{b}{2}\right)}\phi_X\left(\frac{bt}{2}\right) \\ &= e^{jt\left(a+\frac{b}{2}\right)}\sin\left(\frac{bt}{2}\right)\Big/\left(\frac{bt}{2}\right)\end{aligned}$$

7.3 Let X = inside sleeve diameter (cm) and Y = rod diameter (cm). Then

$$\begin{aligned}f_X(x) &= \frac{1}{0.04}\ , \quad 1.98 \le x \le 2.02 \\ &= 0\ , \quad \text{elsewhere}\end{aligned}$$

$$\begin{aligned}f_Y(y) &= \frac{1}{0.05}\ , \quad 1.95 \le y \le 2.00 \\ &= 0\ , \quad \text{elsewhere}\end{aligned}$$

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

(a) $$\begin{aligned}P(Y < X) &= \frac{\text{shaded area}}{\text{total area}} \\ &= 1 - \frac{(0.02)(0.02)/2}{(0.04(0.05)} \\ &= 0.9\end{aligned}$$

(b) $$\begin{aligned}P(X - Y \ge 0.02) &= \frac{\text{shaded area}}{\text{total area}} \\ &= 1 - \frac{(0.04)(0.04)/2}{(0.04(0.05)} \\ &= 0.6\end{aligned}$$

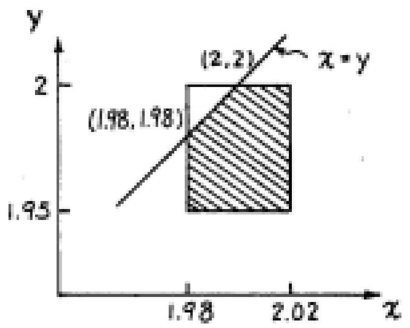

Figure 7.3a

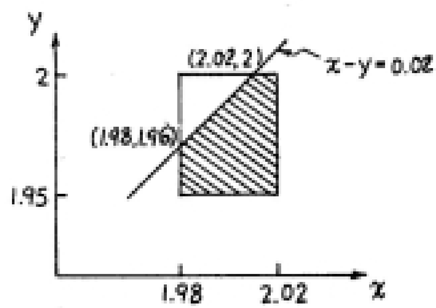

Figure 7.3b

7.4 $f_X(x) = \frac{1}{\sqrt{2\pi}(0.02)} e^{-(x-2)^2/2(0.02)^2}$, $-\infty \le x \le \infty$

$f_Y(y) = \frac{1}{0.05}$, $1.95 \le y \le 2.00$

$= 0$, elsewhere

$f_{XY}(x,y) = f_X(x) f_Y(y)$

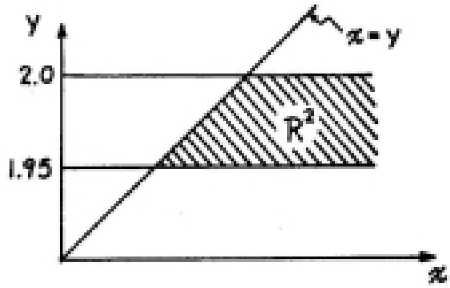

Figure 7.4

(a) $P(Y < X) = \iint_{R^2} f_{XY}(x,y)dxdy$

$$= \int_{1.95}^{2.00}\int_{1.95}^{x} f_{XY}(x,y)dydx + \int_{2.00}^{\infty}\int_{1.95}^{2.00} f_{XY}(x,y)dydx$$

The first integral gives

$$\frac{1}{0.05}\int_{1.95}^{2.00}\int_{1.95}^{x} f_X(x)dydx = \frac{1}{0.05}\int_{1.95}^{2.00} f_X(x)(x-1.95)dx$$

$$= \frac{1}{0.05}\int_{1.95}^{2.0}(x-2)f_X(x)dx + \int_{1.95}^{2.00} f_X(x)dx$$

$$= \frac{1}{0.05\sqrt{2\pi}(0.02)}\int_{-0.05}^{0} ue^{-u^2/0.0008}du + F_X(2.0) - F_X(1.95)$$

$$= \frac{-0.0008/2}{0.05\sqrt{2\pi}(0.02)} e^{-u^2/0.0008}\Big|_{-0.05}^{0} + F_U(0) - F_U\left(\frac{1.95-2.0}{0.02}\right)$$

$= -0.1526 + (0.5 - 1 + 0.9938)$

$= 0.3412$

The second integral $= 0.5$ by inspection. Hence,

$$P(Y < X) = 0.3412 + 0.5 = 0.8412$$

(b) $P(X - Y \ge 0.02) = 0.579$ by using similar procedure.

7.5 Equation (6.10) shows that a binomial-distributed r.v. X can be represented by

$X = X_1 + X_2 + \ldots + X_n$

where X_j, $j = 1, 2, \ldots, n$ are independent and identically distributed with $P(X_j = 1) = p$ and $P(X_j = 0) = q(p + q = 1)$. The mean and variance of X are

$m_X = np$, $\sigma_X^2 = npq$

Based upon the Central Limit Theorem, we immediately deduce from the above that, as $n \to \infty$, r.v. U defined by

$$U = \frac{X - np}{\sqrt{npq}}$$

approaches $N(0,1)$.

Consider, for example, a binomially distributed r.v. X with $n = 15$ and $p = 0.4$. We wish to calculate $P(X = 4)$.

$$P(X = 4) = \binom{15}{4}(0.4)^4(0.6)^{15-4} = 0.127$$

Using normal approximation, we have

$$\begin{aligned} P(X=4) &\cong P\left(\frac{3.5-6}{1.897} < U \leq \frac{4.5-6}{1.897}\right) \\ &= F_U(-0.791) - F_U(-1.318) \\ &= F_U(1.318) - F_U(0.791) \\ &= 0.9063 - 0.7855 \\ &= 0.1208 \end{aligned}$$

7.6 (a) $f_T(450) = \frac{1}{\sqrt{2\pi}40} e^{-\frac{(450-400)^2}{3200}} = 4.566 \times 10^{-3}$

(b) $P(T \leq 450) = F_T(450) = F_U\left(\frac{450-400}{40}\right) = 0.8944$

(c)
$$\begin{aligned} P(|T-m_T| \leq 20) &= P(380 \leq T \leq 420) = F_T(420) - F_T(380) \\ &= F_U\left(\frac{420-400}{40}\right) - F_U\left(\frac{380-400}{40}\right) \\ &= F_U(0.5) - [1 - F_U(0.5)] \\ &= 0.383 \end{aligned}$$

(d)
$$\begin{aligned} P(|T-m_T| \leq 20 | T \geq 300) &= \frac{P(|T-m_T| \leq 20 \cap T \geq 300)}{P(T \geq 300)} \\ &= \frac{P(|T-m_T| \leq 20)}{P(T \geq 300)} = \frac{0.383}{1 - F_U\left(\frac{300-400}{40}\right)} \\ &= \frac{0.383}{F_U(2.5)} = \frac{0.383}{0.9938} = 0.3854 \end{aligned}$$

7.7 It suffices to show that

$$E\{|X|\} = \sqrt{2/\pi}\sigma$$

for a zero-mean normally distributed r.v. X.

$$\begin{aligned} E\{|X|\} &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} |x| e^{-x^2/2\sigma^2} dx = \frac{2}{\sqrt{2\pi}\sigma} \int_0^{\infty} x e^{-x^2/2\sigma^2} dx \\ &= \frac{2}{\sqrt{2\pi}\sigma} \left[-\sigma^2 e^{-x^2/2\sigma^2}\right]_0^{\infty} = \sqrt{2/\pi}\sigma \end{aligned}$$

7.8 Let $W = X + Y$ and $Z = X - Y$. Then W and Z are normal and we only need to show that they are uncorrelated.

$$\begin{aligned} \mu_{11} &= E\{(W-m_W)(Z-m_Z)\} = E\{[(X-m_X)+(Y-m_Y)][(X-m_X)-(Y-m_Y)]\} \\ &= \sigma_X^2 - \sigma_Y^2 = 0 \quad \text{if } \sigma_X^2 = \sigma_Y^2 \end{aligned}$$

7.9 (a) Consider the probabilities $P(X_1 \geq 45)$ and $P(X_2 \geq 45)$.

$$P(X_1 \geq 45) = 1 - F_{X_1}(45) = 1 - F_U\left(\frac{45-40}{6}\right) = 1 - 0.797 = 0.203$$

$$P(X_2 \geq 45) = 1 - F_{X_2}(45) = 1 - F_U\left(\frac{45-45}{3}\right) = 1 - 0.5 = 0.5$$

Hence, X_2 is preferred.

(b) $P(X_1 \geq 48) = 0.092$
$P(X_2 \geq 48) = 0.159$

X_2 is still preferred.

7.10 It suffices to show that

$$\begin{aligned} \alpha_n &= 0 \ , \qquad n \text{ odd} \\ &= 1(3)\cdots(n-1)\sigma^n \ , \ n \text{ even} \end{aligned}$$

for a zero-mean normally-distributed r.v. X. From Eq. (7.12) we have

$$\phi_X(t) = e^{-\sigma^2 t^2/2}$$

Since $\alpha_n = j^{-n}\phi_X^{(n)}(0)$, we have $\alpha_1 = 0$, $\alpha_2 = \sigma^2$, $\alpha_3 = 0$, $\alpha_4 = 3\sigma^4$, etc., and Eq. (7.13) follows.

7.11 Equation (7.38) gives

$$\begin{aligned} \phi_{\underline{X}}(\underline{t}) &= \exp\left[-\frac{1}{2}\sum_{i,j=1}^{n}\lambda_{ij}t_it_j\right] \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^k k!}\left(\sum_{i,j=1}^{n}\lambda_{ij}t_it_j\right)^k \end{aligned}$$

Now

$$E\{X_1X_2X_3\} = j^{-3}\left[\frac{\partial^3}{\partial t_1 \partial t_2 \partial t_3}\phi_{\underline{X}}(\underline{t})\right]_{\underline{t}=0}$$

Upon partial differentiation, all terms in the above expansion vanish for $2k < 3$ and, upon letting $\underline{t} = 0$, all terms disappear for which $2k > 3$. Hence, $E\{X_1X_2X_3\} = 0$.

For $E\{X_1X_2X_3X_4\}$, we see from the above argument that the only contributing term is that for which $2k = 4$ or $k = 2$. We have

$$\begin{aligned} E\{X_1X_2X_3X_4\} &= j^{-4}\left[\frac{\partial^4}{\partial t_1 \partial t_2 \partial t_3 \partial t_4}\phi_{\underline{X}}(\underline{t})\right]_{\underline{t}=0} \\ &= E\{X_1X_2\}E\{X_3X_4\} + E\{X_1X_3\}E\{X_2X_4\} + E\{X_1X_4\}E\{X_2X_3\} \end{aligned}$$

7.12 Let X be the total length.

(a) X is normal with

$m_X = 4 + 4 = 8$ in

$\sigma_X^2 = 0.02 + 0.02 = 0.04$ in^2

(b) $P(7.9 \leq X \leq 8.1)$

$$\begin{aligned} &= F_X(8.1) - F_X(7.9) \\ &= F_U\left(\frac{8.1-8}{0.2}\right) - F_U\left(\frac{7.9-8}{0.2}\right) \\ &= F_U(0.5) - [1 - F_U(0.5)] \\ &= 2(0.6915) - 1 = 0.383 \end{aligned}$$

7.13 Let $Y = X_1+X_2+X_3$. Y is normal with $m_Y = m_{X_1}+m_{X_2}+m_{X_3} = 3$ and $\sigma_Y^2 = \sigma_{X_1}^2+\sigma_{X_2}^2+\sigma_{X_3}^3 = 3$. Hence,

$$P(X_1 + X_2 + X_3 > 1) = P(Y > 1) = 1 - F_Y(1)$$
$$= 1 - F_U\left(\frac{1-3}{\sqrt{3}}\right) = 0.876$$

7.14 Let X_j , $j = 1, 2, \ldots, 100$, be the breaking strength of the jth strand and let $Y = X_1 + X_2 + \cdots + X_{100}$. Then, based on the Central Limit Theorem, Y is approximately normal with $m_Y = (100)(20) = 2000$ and $\sigma_Y^2 = (100)(16) = 1600$. Hence,

$$P(Y \geq 2100) = 1 - F_Y(2100) = 1 - F_U\left(\frac{2100-2000}{40}\right)$$
$$= 1 - F_U(2.5) = 1 - 0.9938$$
$$= 0.0062$$

7.15 $$m_Y = c_1 m + c_2 m + \cdots + c_n m = m \sum_{j=1}^{n} c_j$$

$$\sigma_Y^2 = c_1^2\sigma^2 + c_2^2\sigma^2 + \cdots + c_n^2\sigma^2 = \sigma^2 \sum_{j=1}^{n} c_j^2$$

In order that $m_Y = m$ and $\sigma_Y^2 = \sigma^2$, we must have $\sum_{j=1}^{n} c_j = 1$ and $\sum_{j=1}^{n} c_j^2 = 1$. Since $\left[\sum_{j=1}^{n} c_j\right]^2 > \sum_{j=1}^{n} c_j^2$ if c_j's are positive, the above requirements cannot be satisfied if all c_j's are positive.

7.16 For $x \leq 0$, $f_X(x) = 0$

For $x > 0$,

$$F_X(x) = p(X \leq x)$$
$$= P(-x < \cup \leq x)$$
$$= F_U(x) - [1 - F_U(x)]$$
$$= 2F_U(x) - 1$$

$$f_X(x) = \frac{dF_X(x)}{dx} = 2f_U(x)$$
$$= \sqrt{\frac{2}{\pi}} e^{-x^2/2}$$

Hence,

$$f_X(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2} \ , \quad x > 0$$
$$= 0 \ , \qquad x \leq 0$$

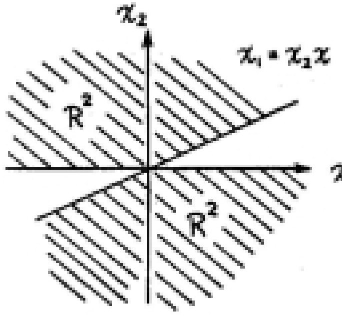

Figure 7.17

7.17 (a) $F_X(x) = \int\int_{R^2: x_1/x_2 \le x} f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2$

$$= \int_0^\infty \int_{-\infty}^{x_2 x} f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2$$
$$+ \int_{-\infty}^0 \int_{x_2 x}^\infty f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2$$
$$= \int_0^\infty F_{X_1}(x_2 x) f_{X_2}(x_2) dx_2 + \int_{-\infty}^0 [1 - F_{X_1}(x_2 x)] f_{X_2}(x_2) d$$

$$f_X(x) = \frac{dF_X(x)}{dx} = \int_{-\infty}^\infty |x_2| f_{X_1}(x_2 x) F_{X_2}(x_2) dx_2$$
$$= \frac{1}{2\pi} \int_{-\infty}^\infty |x_2| e^{-x_2^2(1+x^2)/2} dx_2$$

From Problem 7.7, we have

$$\frac{\sqrt{(1+x^2)}}{\sqrt{2\pi}} \int_{-\infty}^\infty |x_2| e^{-x_2^2(1+x^2)/2} dx_2 = \sqrt{\frac{2}{\pi(1+x^2)}}$$

Hence,

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty \le x \le \infty$$

(b) See the solution to Problem 4.2(c).

7.18 A solution is given in Problem 5.29.

7.19 $Y = e^X$ where X is $N(m, \sigma^2)$.

$$E\{Y\} = E\{e^X\} = \phi_X(t)\big|_{jt=1} = e^{jmt - \sigma^2 t^2/2}\big|_{jt=1}$$
$$= e^{m+\sigma^2/2} = \theta_Y \exp(\sigma_{\ln Y}^2/2)$$

$$E\{Y^2\} = E\{e^{2X}\} = \phi_X(t)\big|_{jt=2} = e^{2m+2\sigma^2} = \theta_Y^2 \exp(2\sigma_{\ln Y}^2)$$

$$\sigma_Y^2 = E\{Y^2\} - E^2\{Y\} = \theta_Y^2 \exp(\sigma_{\ln Y}^2)\left[\exp(\sigma_{\ln Y}^2) - 1\right]$$
$$= m_Y^2[\exp(\sigma_{\ln Y}^2) - 1]$$

7.20 (a) The parameters θ_X and $\sigma_{\ln X}^2$ are found from

$1 = \theta_X \exp(\sigma_{\ln X}^2/2)$

$0.09 = 1^2[\exp(\sigma_{\ln X}^2) - 1]$

which gives

$\theta_X = 0.958$, $\sigma_{\ln X}^2 = 0.086$

$$P(X > 1.2) = 1 - F_X(1.2) = 1 - F_U\left[\frac{\ln(1.2/0.958)}{\sqrt{0.086}}\right] = 0.221$$

(b) Y is lognormal distributed with $m_Y = a + bm_X = a + b$ and $\sigma_Y^2 = b^2\sigma_X^2 = 0.09b^2$. Hence,

$$f_Y(y) = f_X\left(\frac{y-a}{b}\right)\left(\frac{1}{b}\right)$$
$$= \frac{1}{0.294\sqrt{2\pi}(y-a)} \exp\left[-\frac{1}{0.172}\ln^2\left(\frac{y-a}{0.958b}\right)\right], y \ge a$$
$$= 0, \qquad \text{elsewhere}$$

7.21 Mean time between arrivals $= \frac{1}{\lambda}$

$$= \frac{1}{20(\text{vph})} = 3\,\text{min}$$

Percentage of time $= \frac{2}{3}$.

7.22 $$f_T(t) = \frac{1}{25}e^{-t/25} \ , \ t \geq 0$$
$$= 0 \ , \qquad \text{elsewhere}$$

$$P(T \geq 35) = \int_{35}^{\infty} f_T(t)dt = e^{-35/25} = 0.247$$

$$P \text{ (at least two still stand)} = \binom{3}{2}(0.247)^2(0.753) + \binom{3}{3}(0.247)^3$$
$$= 0.153$$

7.23 $$f_X(x) = \frac{\lambda^\eta}{\Gamma(\eta)}x^{\eta-1}e^{-\lambda x} \ , \ x \geq 0$$
$$= 0 \ , \qquad \text{elsewhere}$$

$$\alpha_k = E\{X^k\} + \frac{\lambda^\eta}{\Gamma(\eta)}\int_0^{\infty} x^{k+\eta-1}e^{-\lambda x}dx$$

Let $\lambda x = y$

$$\alpha_k = \frac{1}{\lambda^k\Gamma(\eta)}\int_0^{\infty} y^{k+\eta-1}e^{-y}dy = \frac{\Gamma(\eta+k)}{\lambda^k\Gamma(\eta)}$$

(a) $$m_X = \alpha_1 = \frac{\Gamma(\eta+1)}{\lambda\Gamma(\eta)} = \frac{\eta}{\lambda}$$

$$\alpha_2 = \frac{\Gamma(\eta+2)}{\lambda^2\Gamma(\eta)} = \frac{\eta(\eta+1)}{\lambda^2}$$

and

$$\sigma_X^2 = \frac{\eta(\eta+1)}{\lambda^2} - \frac{\eta^2}{\lambda^2} = \frac{\eta}{\lambda^2}$$

(b) $$\alpha_3 = \frac{\Gamma(\eta+3)}{\lambda^3\Gamma(\eta)} = \frac{\eta(\eta+1)(\eta+2)}{\lambda^3}$$

$$\gamma_1 = \mu_3/\sigma^3 = (\alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^2)/\sigma^3 = 2\sqrt{\eta} > 0$$

7.24 For $x > 0$,

$$F_X(x) = \int_0^x \frac{\lambda^\eta}{(\eta-1)!}u^{\eta-1}e^{-\lambda u}du = 1 - \int_x^{\infty} \frac{\lambda}{(\eta-1)!}(\lambda u)^{\eta-1}e^{-\lambda u}du$$

Let $y = \lambda u$. The desired result is obtained by successive integration by parts.

7.25 The r.v. T has a gamma distribution with $\eta = 3$ and $\lambda = 0.01$. Thus,

$$f_T(t) = \frac{(0.01)^3}{2!}t^2e^{-0.01t} \ , \ t \geq 0$$
$$= 0 \ , \qquad \text{elsewhere}$$

Using the results obtained in Problem 7.24, we have

$$P(T \geq 300) = 1 - F_T(300) = e^{-(0.01)(300)}\left(1 + 3 + \frac{3^2}{2}\right) = 0.423$$

7.26 Let $h(t) = \lambda$. Then $P(t < T \le t + dt | T \ge t) = \lambda dt$. Now

$$P(t < T \le t + dt \big| T \ge t) = \frac{P(t < T \le t + dt \cap T \ge t)}{P(T \ge t)}$$

$$= \frac{P(t < T \le t + dt)}{P(T \ge t)} = \frac{f_T(t)dt}{1 - F_T(t)}$$

Hence,

$$\frac{f_T(t)dt}{1 - F_T(t)} = \lambda dt$$

or $f_T(t) = \lambda \left[1 - \int_0^t f_T(t)dt\right]$

Upon differentiating both sides of the above, we have

$$\frac{df_T(t)}{dt} + \lambda f_T(t) = 0 \ , \ t \ge 0$$

With the condition $\int_0^\infty f_T(t) = 1$, the solutions to the differential equation is

$$f_T(t) = \lambda e^{-\lambda t} \ , \ t \ge 0$$

7.27 $Y = a + X$, $g(x) = a + x$ and $g^{-1}(y) = y - a$.

We have

$$f_Y(y) = f_X\left[g^{-1}(y)\right]\left|\frac{dg^{-1}(y)}{dy}\right| = \lambda e^{-\lambda(y-a)} \ , \ y \ge a$$

$$= 0 \ , \qquad \text{elsewhere}$$

Since $Y = a + X$, we have

$$m_Y = a + m_X = a + 1/\lambda$$

$$\sigma_Y^2 = \sigma_X^2 = 1/\lambda^2$$

7.28 Equation (7.28) gives $X = U_1^2 + U_2^2 + \cdots + U_n^2$, where U_j, $j = 1, 2, \ldots, n$, are $N(0,1)$. Let $X_j = U_j^2$. Then $E\{X_j\} = E\{U_j^2\} = 1$ and, as seen from Problem 7.11,

$$E\{X_j^2\} = E\{U_j^4\} = 3E^2\{U_j^2\} = 3$$

$$\sigma_{X_j}^2 = 3 - 1 = 2$$

Hence, $m_X = n$ and $\sigma_X^2 = 2n$.

It thus follows from the Central Limit Theorem that $(X - n)/\sqrt{2n}$ tends to $N(0,1)$ as $n \to \infty$.

7.29 $F_{X_{(j)}}(x) = P[\text{at least } j \text{ of the } X\text{'s} \le x]$

$$= \sum_{k=j}^{n} P[\text{exactly } k \text{ of the } X\text{'s} \le x]$$

$$= \sum_{k=j}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

where $p = P(X_j \le x) = F_X(x)$. Hence,

$$F_{X_{(j)}}(x) = \sum_{k=j}^{n} \binom{n}{k} [F_X(x)]^k [1 - F_X(x)]^{n-k} \ , \ j = 1, 2, \ldots, n$$

7.30 From Problem 7.29, we have $(n = 10)$,

(a) $P(X_{(10)} \le 3/4) = F_{X_{(10)}}(3/4) = [F_X(3/4)]^{10}$

Since $F_X(x) = x$ for $0 \le x \le 1$, $F_X(3/4) = 3/4$, and

$$P(X_{(10)} \le 3/4) = (3/4)^{10} = 0.056$$

(b) $P(X_{(9)} > 0.5) = 1 - F_{X_{(9)}}(0.5)$

$$= 1 - \left\{ \binom{10}{9} [F_X(0.5)]^9 [1 - F_X(0.5)] + \binom{10}{10} [F_X(0.5)]^{10} \right\}$$

$$= 0.989$$

7.31 As is seen from the solution to Problem 7.30,

$$P_0(0, t) = \exp(-t^v/w) \ , \ t \ge 0$$

Hence,

$$F_T(t) = 1 - p_0(0, t) = 1 - \exp(-t^v/w) \ , \ t \ge 0$$

and

$$f_T(t) = \frac{dF_T(t)}{dt} = (v/w)t^{v-1} \exp(-t^v/w) \ , \ t \ge 0$$

$$= 0 \ , \qquad \text{elsewhere}$$

7.32 The structural system is one with (initially) n components in parallel. Let $q_{nk}(s)$ be the probability that failure will occur to $n - k$ among n initially existing members.

Consider first $q_{nn}(s)$ (probability of no failure). Clearly,

$$q_{nn}(s) = [P(R > s/n)]^n = [1 - F_R(s/n)]^n$$

For $k = 0, \ 1, 2, \ldots, \ n - 1$, we have

$$q_{nk}(s) = \binom{n}{1} F_R\left(\frac{s}{n}\right) p^n_{(n-1)k}(s) + \binom{n}{2} \left[F_R\left(\frac{s}{n}\right)\right]^2 p^n_{(n-2)k}(s) + \cdots + \binom{n}{n-k} \left[F_R\left(\frac{s}{n}\right)\right]^{n-k} p^n_{kk}(s) \qquad (1)$$

where

$p^i_{jk}(s) =$ the probability that failure occurs to $j - k$ members among currently existing j members with resisting strength greater than s/i, thus reducing the number of remaining members from j to k.

We then have

$$p^n_{kk}(s) = [1 - F_R(s/n)]^k$$

$$p^i_{jk}(s) = \binom{j}{1} [F_R(s/j) - F_R(s/i)] p^j_{(j-1)k}(s) + \binom{j}{2} [F_R(s/j) - F_R(s/i)]^2 p^j_{(j-2)k}(s) + \cdots + \binom{j}{j-k} [F_R(s/j) - F_R(s/i)]^{j-k} p^j_{kk}(s) \ , \ n \ge i \ge j > k$$

The substitution of the above into Eq. (1) gives the desired result.

7.33 Let q_{nk} be the desired probability. Then

$$q_{nk} = \int_0^\infty q_{nk}(s) f_S(s) ds$$

where $q_{nk}(s),\ k = 0, 1, 2, \ldots, n$, are given in Problem 7.32.

7.34 For $n = 3$, we obtain

$$\begin{aligned}
q_{33}(s) &= [1 - F_R(s/3)]^3 \\
q_{32}(s) &= 3F_R(s/3)[1 - F_R(s/2)]^2 \\
q_{31}(s) &= \{6F_R(s/3)[F_R(s/2) - F_R(s/3)] + 3[F_R(s/3)]^2\}[1 - F_R(s)] \\
q_{30}(s) &= 3F_R(s/3)[F_R(s/2) - F_R(s/3)]\{2[F_R(s) - F_R(s/2)] + [F_R(s/2) \\
&\quad - F_R(s/3)]\} + 3[F_R(s/3)]^2[F_R(s) - F_R(s/3)] + [F_R(s/3)]^3
\end{aligned}$$

For this problem, $s = 270$ and

$$\begin{aligned}
F_R(r) &= 0\ , && r < 80 \\
&= r/20 - 4\ , && 80 \le r \le 100 \\
&= 1\ , && r > 100
\end{aligned}$$

Hence,

$$\begin{aligned}
F_R(s) &= F_R(270) = 1 \\
F_R(s/2) &= F_R(135) = 1 \\
F_R(s/3) &= F_R(90) = 90/20 - 4 = 0.5
\end{aligned}$$

and

$$\begin{aligned}
q_{33}(s) &= (1 - 0.5)^3 = 0.125 \\
q_{32}(s) &= 0 \\
q_{31}(s) &= 0 \\
q_{30}(s) &= 3(0.5)^3 + 3(0.5)^3 + (0.5)^3 = 0.875
\end{aligned}$$

Hence, it is seen that the structure completely fails with probability 0.875 and is safe with probability 0.125. No partial failure is possible.

7.35 Required graphs are easily plotted from Eqs. (7.123) and (7.124).

7.36 We see from Eqs. (7.89) and (7.91) that

$$F_Y(y) = [F_X(y)]^n \quad \text{and} \quad F_Z(z) = 1 - [1 - F_X(z)]^n$$

Let us first determine $f_{YZ}(y, z)$. We write

$$\begin{aligned}
P(Y \le y) &= P(Y \le y \cap Z \le z) + P(Y \le y \cap Z > z) \\
&= F_{YZ}(y, z) + P(Y \le y \cap Z > z)
\end{aligned}$$

or

$$F_{YZ}(y, z) = [F_X(y)]^n - P(Y \le y \cap Z > z)$$

But

$$P(Y \le y \cap Z > z) = P[z < X_1 \le y \cap z < X_2 \le y \cap \cdots \cap z < X_n \le y]$$
$$= \prod_{j=1}^{n} P[z < X_j \le y]$$

Clearly,

$$P(z < X_j \le y) = 0 \ , \ z \ge y$$
$$= F_X(y) - F_X(z) \ , \ z < y$$

Hence,

$$F_{YZ}(y,z) = [F_X(y)]^n \ , \ z \ge y$$
$$= [F_X(y)]^n - [F_X(y) - F_X(z)]^n \ , \ z < y$$

and

$$f_{YZ}(y,z) = \frac{\partial F_{YZ}(y,z)}{\partial y \partial z} = 0 \ , \ z \ge y$$
$$= n(n-1)[F_X(y) - F_X(z)]^{n-2} f_X(y) f_X(z) \ , z < y$$

Now consider S.

$$F_S(s) = P(S \le s) = P(Y - Z \le s)$$
$$= \iint_{R^2 : y-z \le s} f_{YZ}(y,z) dy dz$$
$$= \int_{-\infty}^{\infty} \int_{y-s}^{\infty} f_{YZ}(y,z) dz dy$$

Hence,

$$F_S(s) = 0 \ , \qquad s < 0$$
$$= \int_{-\infty}^{\infty} \int_{y-s}^{y} n(n-1)[F_X(y) - F_X(z)]^{n-2} f_X(y) f_X(z) dz dy \ , \ s \ge 0$$

and

$$f_S(s) = \frac{dF_S(s)}{ds} = 0 \ , \qquad s < 0$$
$$= n(n-1) \int_{-\infty}^{\infty} [F_X(y) - F_X(y-s)]^{n-2} f_X(y-s) f_X(y) dy \ , \ s \ge 0$$

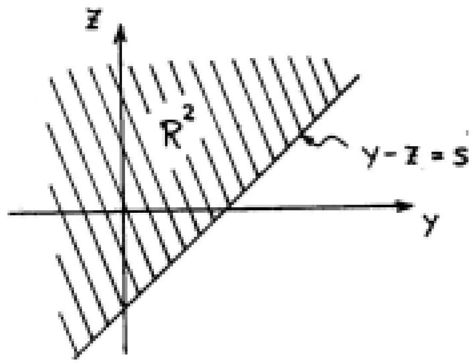

Figure 7.36